

Time Dependent Interface in AdS Black Hole Spacetime

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Abstract

We consider a D5-brane solution in AdS black hole spacetime. This is a defect solution moving in subspace of $AdS_5 \times S^5$. This non-local object is realized by the probe D5-brane moving in black hole spacetime. Here we consider the D5-brane which attaches at two points on the AdS boundary which are interpreted as two interfaces in the boundary gauge theories. We found this probe brane does not reach the black hole horizon and its configuration is limited to the outside of the horizon. We also found the solution does not depend on the motion on S^5 subspace.

Keywords: Quantum; AdS black hole; Angular velocity

Introduction

In the AdS/CFT correspondence non-local operators are a useful tool for studying [1-3]. For example, in [4], we considered a kind of non-local operator called “an interface” and found the agreement in the results of gauge theory and gravity theory in the leading order of the power series of λ/k^2 . Here λ is the 't Hooft coupling $\lambda = g_{YM}^2 N$ and k is a parameter which specifies the gauge flux on the probe brane. This non-local operator is realized in the D3/D5 brane system. The multiple D3-brane create a curved space-time and one D5-brane is treated as a probe on the $AdS_5 \times S^5$ spacetime. These brane systems are related to non-equilibrium systems in the AdS/CFT correspondence [5-8]. A time dependent solution of the D5-brane we consider in this paper has its application to cosmology in anti-de-Sitter space and brane world scenario [9,10].

The motivation to consider the AdS/CFT correspondence including the D5-brane on black hole spacetimes is related to a conjecture called “Complexity-Action” (CA) [11,12]. In this statement complexity is a quantity which is expected to be related many black hole problems [13,14]. Complexity has the origin in computational science and it quantifies how hard to create the final state from the initial state [15-21]. In black hole physics this quantity describes a quantum state of the black hole interior or Einstein-Rosen Bridge [22]. According to the holographic relation stated in the first paragraph, complexity must have the holographic counterpart. That is called holographic complexity. CA conjecture [23,24] states that this holographic quantity is the action which is calculated in a bulk region called “The Wheeler DeWitt Patch”. This is the region bounded by null surfaces anchored at the given time on boundaries. The calculation of the action has a difficulty caused by the

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divergence at the AdS boundary where the metric diverges. To compute the growth rate of the action, we only need to take into account the bulk region inside the horizon as explained in [25]. The left panel of **FIG. 1** depicts Wheeler DeWitt patch on a Penrose diagram. The two diagonal lines represent the black hole horizon. As the time passes, this region changes as shown in the right of **FIG. 1**. The time defined on a boundary theory is denoted by t_L . The separated regions (2), (3) and (4) do not contribute the development. We only have to find the contribution of region (1) to measure the time development of the WDW action.

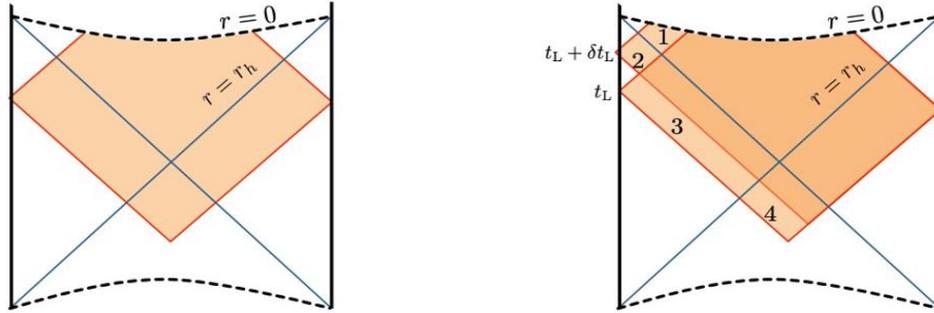


FIG. 1. Left: Wheeler DeWitt patch. Right: Development of the region.

We found the static defect solutions in the AdS black hole spacetime in [26, 27]. These describe the embedding of the D5-brane in AdS black hole spacetime. The results tell us that the D5-brane can only extend outside of the horizon if there is a non-zero-gauge flux on the D5-brane. In CA conjecture holographic complexity, the action is calculated by the integral over the Wheeler-DeWitt patch. Especially, in order to find the growth rate of it, we only have to integrate inside the horizon. Then the DBI action does not contribute the growth of complexity in these cases. A time dependent system is interesting for studying the AdS/CFT correspondence. Its behavior is different from the static cases. For example, the extremal surfaces do not reach the horizon in a static case, whereas it can penetrate the horizon in the time dependent case as stated in [28]. Then it will be worth to consider a moving defect [29] in AdS black hole spacetime. These time dependent cases have never been studied for the AdS/CFT correspondence and CA conjecture. If such a solution is found, this can be a good way to test CA relation. As stated in the first, a new parameter is useful to compare the holographic quantities. In the above reason we would like to consider a moving object in the AdS black hole spacetime. The whole spacetime consists of the product of AdS_5 and S^5 . We can suppose some types of motion of the D-branes. For simplicity, in this paper we would like to consider a motion which has a spherical symmetry which has the rotation in S^2 subspace of S^5 . This D5-brane touches the AdS boundary at least on the one side, which realizes the interface on the boundary gauge theory, extend toward the inside of the AdS bulk space and may or may not reach the inside of the horizon on the other side. If the D5-brane does not reach the black hole horizon and came back to the boundary, this configuration represents the situation where there are two interfaces on the boundary gauge theory.

The organization of this paper is as follows. In Section 2 we start with the study of the flat AdS case and find its solution. This case corresponds to the flat spacetime solution [4] added the angular velocity. In Section 3 we generalize the solution to the black hole spacetime. We conclude this paper in Section 4 with some comments for the results.

Flat Spacetime

In this section we consider the embedding of the D5-brane in AdS spacetime $AdS_5 \times S^5$. The metric of the AdS part with radius R_{AdS} is given by time coordinate t , Cartesian coordinates $x_i; i = 1, 2, 3$ and the radial coordinate $1/y$ as

$$ds_{AdS}^2 = \frac{R_{AdS}^2}{y^2} (-dt^2 + dy^2 + \sum_{i=1}^3 dx_i^2) \quad (1)$$

The S^5 metric is given by coordinates $\theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$, $\psi \in [0, \pi/2]$ and $\varphi, \chi \in [0, 2\pi]$ as

$$\begin{aligned} ds_{S^5}^2 &= d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2 \\ &= d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta (d\psi^2 + \sin^2\psi d\varphi^2 + \cos^2\psi d\chi^2) \end{aligned} \quad (2)$$

In the following we put $R_{AdS}=1$ for simplicity. The Ramond-Ramond 4-form is defined as

$$C_4 = -\frac{1}{y^4} dt \times dx_1 dx_2 dx_3 + 4\alpha_4 \quad (3)$$

where α_4 is a 4-form satisfying $d\alpha_4 = \text{volume of } S^5$.

We consider the D5-brane rotates in subspace of S^5 . We assume that the D5-brane extends to the directions

$$t, x_1, x_2 \in AdS_5, \psi, \varphi \in S^5, \quad (4)$$

and one dimensional subspace in other directions. This embedding is expressed by one parameter σ . Furthermore we assume that the D5-brane rotates on the boundary at the constant angular velocity as in the case of the string considered in [30]. The edge of the D5-brane sweeps an S^2 subspace of S^5 where χ takes a constant value π . From the above discussion, we write this embedding by the following functions

$$y = y(\sigma), x_3 = x_3(\sigma), \theta = \theta(\sigma), \phi = \omega t + g(\sigma), \quad (5)$$

and χ takes a constant value π . There is a gauge flux on the D5-brane which is expressed by symmetry as

$$F = F^j(\psi) d\psi \wedge d\varphi, A = F(\psi) d\varphi. \quad (6)$$

In this section ' means the derivative by the variables indicated in eq.(5) and eq.(6). The induced metric is

$$\begin{aligned} ds_{D5}^2 &= -\left(\frac{1}{y^2} - \omega^2 \sin^2\theta\right) dt^2 + \left(\frac{y'^2 + x_3'^2}{y^2} \theta'^2 + g'^2 \sin^2\theta\right) d\sigma^2 + 2\omega g' \sin^2\theta dt d\sigma \\ &\quad + \frac{dx_1^2 + dx_2^2}{y^2} + \cos^2\theta (d\psi^2 + \sin^2\psi d\varphi^2) \end{aligned}$$

Adding the flux, the above metric is in matrix form

$$G_{ind} + F = \left[\begin{array}{ccc} -\frac{1}{y^2}(1-\omega^2 y^2 \sin^2 \theta) & \omega g' \sin^2 \theta & \\ \frac{\omega g' \sin^2 \theta}{\omega g' \sin^2 \theta} & \frac{1}{y^2}(y'^2 + x_3'^2 + y^2 \theta'^2 \sin^2 \theta) & \\ & & \frac{1}{y^2} \\ & & \frac{1}{y^2} \\ & & \frac{\cos^2 \theta}{F'} \frac{F'}{\cos^2 \theta \sin^2 \psi} \end{array} \right] \quad (7)$$

Then the DBI action is

$$S_{DBI} / T_5 = -2\pi T \int V d\sigma d\psi \frac{\sqrt{\cos^4 \theta \sin^2 \psi + F'^2}}{y^4} L(\sigma) \quad (8)$$

where T is the integral over the time interval and V is the integral on (x1, x2) plane. The Wess-Zumino action is, by substituting the RR4-form (3),

$$S_{WZ} / T_5 = \int F \wedge C_4 = -2\pi T V \int \frac{F' x_3' d\psi d\sigma}{y^4} \quad (9)$$

Summing them, the D5-brane action is

$$S_{D5} / T_5 = -2\pi T V \int \frac{d\psi d\sigma}{y^4} \left(\sqrt{\cos^4 \theta \sin^2 \psi} \right) + F'(\psi)^2 L(\sigma) + F'(\psi) x_3' \quad (10)$$

The equation of motion for F(ψ) is

$$\frac{d}{d\psi} \left(\frac{F'(\psi)}{\sqrt{\cos^4 \theta \sin^2 \psi + F(\psi)^2}} \right) = 0 \quad (11)$$

By using c(σ), which is a function of σ,

$$F'^2 = \frac{c^2 \cos^4 \theta}{1 - c^2} \sin^2 \psi, \quad F' = -\kappa \sin \psi \quad (12)$$

where we denote the constant factor in the front of sin ψ as κ since we know F' is independent of y:

$$\frac{c(\sigma)^2 \cos^4 \theta(\sigma)}{1 - c(\sigma)^2} =: \kappa^2 \quad (13)$$

This constant represents the strength of the gauge flux. Therefore we found the ansatz for the gauge flux to be proportional to the volume form of S² subspace:

$$F = -\kappa \sin\psi d\psi \wedge d\varphi \tag{14}$$

The above form of the gauge flux is the form we can expect from the symmetry of the system-rotation symmetry for ψ and φ . Summarizing the above discussion, the action is

$$S_{DS} / T_5 = -2\pi TV \int \frac{d\sigma}{y^4} \left(\sqrt{\cos^4\theta + \kappa^2 L(\sigma)} - \kappa x_3' \right) \tag{15}$$

$$L(\sigma) := \sqrt{(y'^2 + x_3'^2 + y'^2\theta'^2)(1 - \omega^2 y^2 \sin^2\theta) + y^2 g'^2 \sin^2\theta} \tag{16}$$

For simplicity we use the following notation in the next section.

$$\Theta := \sqrt{\cos^4\theta + \kappa^2}, S := y'^2 + x_3'^2 + y'^2\theta'^2, \Omega := 1 - \omega^2 y^2 \sin^2\theta \tag{17}$$

By this notation the Lagrangian is

$$L = \frac{\Theta L(\sigma) - \kappa x_3'}{y^4}, L(\sigma) = \sqrt{S\Omega + y^2 g'^2 \sin^2\theta} \tag{18}$$

Equations of motion

The equations of motion are

$$\frac{d}{d\sigma} \left(\frac{y' \Theta \Omega}{y^4 L} \right) + \frac{4}{y^5} (\Theta L - \kappa x_3') - \frac{\Theta}{y^3} \frac{\theta'^2 \Omega + (g'^2 - \omega^2 S) \sin^2\theta}{L} = 0 \tag{19a}$$

$$\frac{d}{d\tau} \left(\theta' \frac{\Omega \Theta}{y^2 L} \right) + \frac{2 \sin\theta \cos^3\theta}{y^4 \Theta} L - \frac{(g'^2 - \omega^2 S) \sin\theta \cos\theta}{y^2} \frac{\Theta}{L} = 0 \tag{19b}$$

$$\frac{d}{d\tau} \left(x_3' \frac{\Omega \Theta}{y^4 L} \right) + \frac{4\kappa y'}{y^5} = 0 \tag{19c}$$

$$\frac{d}{d\tau} \left(g' \frac{\sin^2\theta \Theta}{y^2 L} \right) = 0 \tag{19d}$$

There is a gauge degree of freedom due to reparametrization invariance in σ . We fix $\Theta/L=1$. Then the equations (19) are

$$\frac{d}{d\sigma} \left(\frac{y' \Omega}{y^4} \right) + \frac{4}{y^5} (\Theta^2 - \kappa v) - \frac{\theta'^2 \Omega + (w^2 - \omega^2 S) \sin^2\theta}{y^3} = 0 \tag{20a}$$

$$\frac{d}{d\sigma} \left(\theta' \frac{\Omega}{y^2} \right) + \left(\frac{2 \sin\theta \cos^3\theta}{y^4} \right) L - \frac{(g'^2 - \omega^2 S) \sin\theta \cos\theta}{y^2} = 0 \tag{20b}$$

$$\frac{d}{d\sigma} \left(x_3' \frac{\Omega}{y^4} \right) + \frac{4\kappa y'}{y^5} = 0 \tag{20c}$$

$$\frac{d}{d\sigma} \left(g' \frac{\sin^2\theta}{y^2} \right) = 0 \tag{20d}$$

We define,

$$A := -\frac{d}{d\sigma} \log \Omega = 2\omega^2 y \sin \theta \frac{(y' \sin \theta + y \theta' \cos \theta)}{\Omega} \quad (21a)$$

$$B := \frac{4}{y\Omega} (\kappa x_3' - \Theta^2), C := \frac{(g'^2 - \omega^2 S)}{\Omega}, D := -\frac{2 \sin \theta \cos^3 \theta y}{2\Omega} \quad (21b)$$

The equations are solved for the second derivative terms as

$$y'' = y\theta'^2 + \frac{4y'^2}{y} + y'A + B + C y \sin^2 \theta \quad (22a)$$

$$\theta'' = \frac{2y'\theta'}{y} + \theta'A + C \sin \theta \cos \theta + D \quad (22b)$$

$$\frac{x_3''}{3} = \frac{4y'x_3'}{y} + x_3'A - \frac{4\kappa y'}{y\Omega} \quad (22c)$$

$$g'' = -2\theta' g' \cot \theta + \frac{2y'g'}{y} \quad (22d)$$

Static Case

In the static case, as we already found in [4], the static configuration of the D5-brane on the flat AdS spacetime is expressed by the embedding $x_3 = \kappa y$ and other coordinates (θ and φ) take constants.

Let us confirm it. For $\omega = 0$ and $\theta = 0$, the equations of motion (19) are simplified by setting $\sigma = y$ as

$$\frac{d}{dy} \left(k \frac{\sqrt{1 + \kappa^2}}{y^4 \sqrt{1 + x_3'^2}} \right) + \frac{4\kappa}{y^5} = 0 \quad (23)$$

Then in this case, the following is the solution:

$$x_3 = \kappa y, \theta(y) = 0, g(y) = 0 \quad (24)$$

This is surely the stationary solution [4].

Boundary condition

$y = 0$ corresponds to the AdS boundary. The boundary condition is, since we fixed $\Theta/L = 1$, in $y \rightarrow 0$ limit

$$1 = \frac{\Theta}{L} = \sqrt{\frac{\cos^4 \theta_0 + \kappa^2}{(y'^2 + x_3'^2 + y^2 \theta'^2)(1 - \omega^2 y^2 \sin^2 \theta_0) + g'(0)^2 y^2 \sin^2 \theta_0}} \rightarrow \sqrt{\frac{\cos^4 \theta_0 + \kappa^2}{y'^2 + x_3'^2}} \quad (25)$$

The righthand side of the third equation (22c) behaves

$$\frac{4y'}{y\Omega} (x_3' (1 - \omega^2 y^2 \sin^2 \theta) - \kappa) \rightarrow \frac{4y'}{y\Omega} (x_3'(0) - \kappa) \quad (26)$$

This includes y in the denominator. So the righthand side of this equation grows infinitely unless the numerator is zero simultaneously.

Then, from eq.(25) and eq.(26) we impose the following condition:

$$y'(0) = \cos^2 \theta_0, x_3'(0) = \kappa \quad (27)$$

We impose the Neumann boundary condition for the angular direction $\theta'(0) = 0$. The righthand side of the forth equation of motion (22d) approaches

$$\frac{2g'}{y \sin\theta} (y' \sin\theta - y\theta' \cos\theta) \rightarrow \frac{2g' \cos\theta_0}{y \sin\theta_0} (\cos\theta_0 \sin\theta_0 - y\theta'),$$

$$\therefore g'(0) = 0 \tag{28}$$

From the above discussion we obtain the appropriate boundary condition

$$y(0) = 0, \theta(0) = \theta_0, x_3(0) = \kappa\sigma, g(0) = 0 \tag{29a}$$

$$y'(0) = \cos^2\theta_0, \theta'(0) = 0, x_3'(0) = \kappa, g'(0) = 0 \tag{29b}$$

Solution

We solve the equations of motion (22) under the initial condition (29). The results are summarized in the following seven figures.

The function y increases linearly for the value of σ in the case $\theta_0 = 0$ as shown in **FIG. 2**.

The next three figures show the gauge flux dependence. **FIG.3** and **FIG.4** show the behavior of θ and g as functions of y . In these figures, neglecting numerical error, we read θ and g do not grow from zero. **FIG. 5** shows linear relationship $x_3 = \kappa y$ as the same in [4].

The next three figures show the angular velocity dependence (see **FIG.6, FIG.7** and **FIG.8**). From these results, we find that there is no dependence on the angular velocity. It can also be seen from the θ plot (**FIG. 6**) and the fact that the ω dependence in equation (20) exists only in the combination with $\sin\theta$.

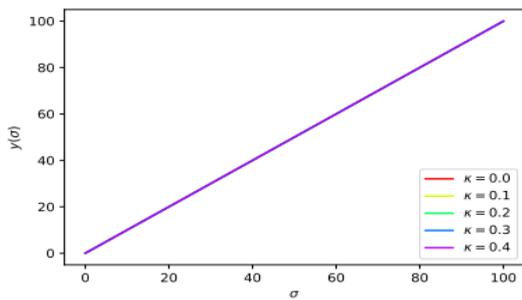


FIG.2. Increase of y for $\theta_0 = 0$.

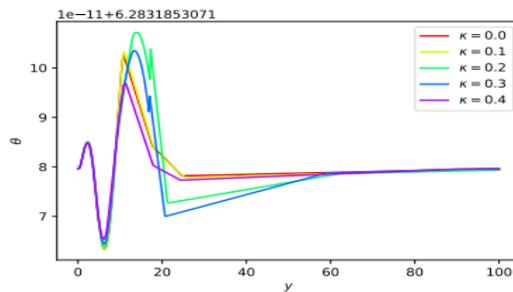


FIG. 3. κ dependence of θ .

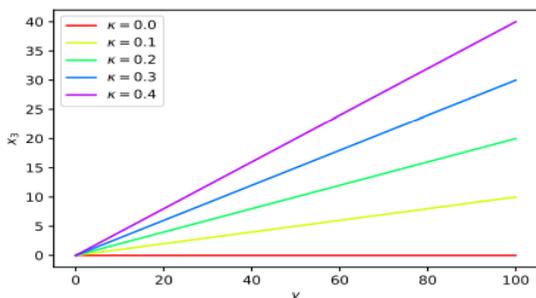


FIG.4. κ dependence of x_3 .

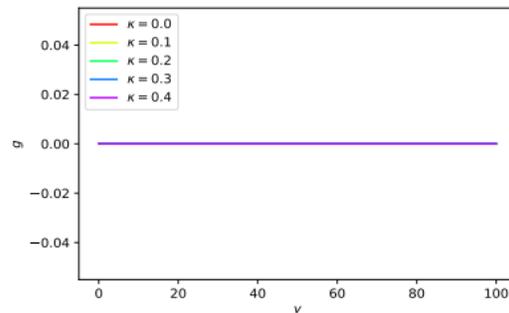


FIG. 5. κ dependence of g .

Black Hole Spacetime

In this section we generalize the solution obtained in the previous section to the black hole spacetime. The metric is

$$ds_{AdS_5}^2 \times S^5 = \frac{r^2}{R_{AdS}^2} \left(-h dt^2 + d\vec{x}^2 \right) + \frac{R_{AdS}^2}{r^2} \frac{dr^2}{h} + R_{AdS}^2 ds_{S^5}^2 \quad (30)$$

$$h(r) = 1 - \frac{r_H^4}{r^4} \quad (31)$$

where R_{AdS} is the radius of the AdS and the S^5 metric is, by coordinates $\theta, \psi \in [0, \pi/2]$ and $\phi, \varphi, \chi \in [0, 2\pi]$,

$$ds_{S^5}^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta (d\psi^2 + \sin^2\psi d\varphi^2 + \cos^2\psi d\chi^2) \quad (32)$$

In the following, we put $R_{AdS} = 1$. By changing variables, $y = 1/r$, this metric becomes

$$ds_{AdS_5 \times S^5}^2 = \frac{1}{y^2} \left(-h dt^2 + \frac{dy^2}{h} + d\vec{x}^2 \right) + ds_{S^5}^2 \quad (33)$$

In the same way as before, we consider the case where the D5-brane extends to the directions

$$t, x_1, x_2 \in AdS_5, \psi, \varphi \in S_5 \quad (34)$$

By parameter σ the embedding of the D5-brane is

$$y = y(\sigma), x_3 = x_3(\sigma), \theta = \theta(\sigma), \phi = \omega t + g(\sigma) \quad (35)$$

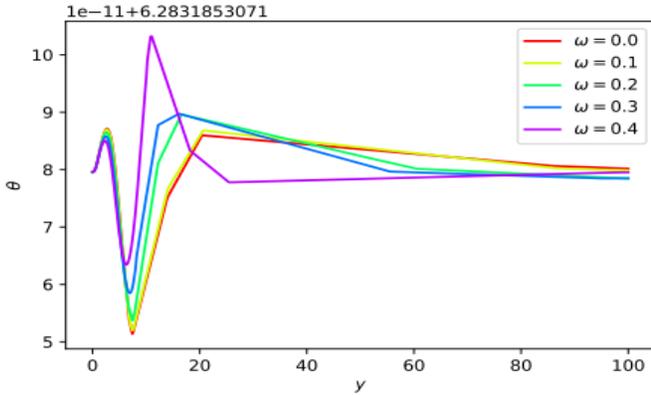


FIG. 6. ω dependence of θ .

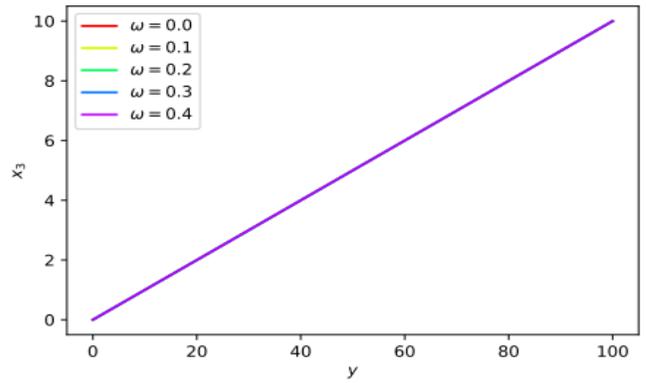


FIG. 7. ω dependence of x_3 .

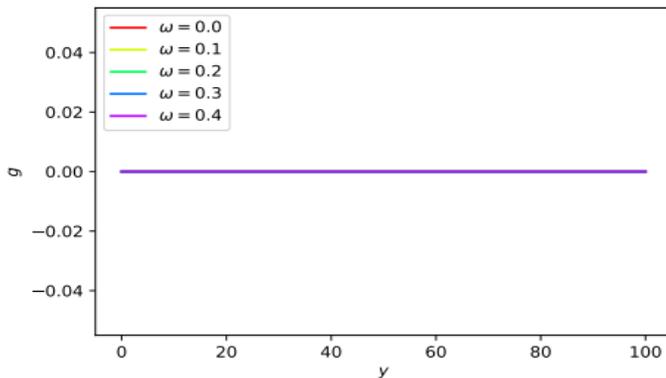


FIG. 8. ω dependence of g .

The gauge flux is obtained in the same way as the flat case,

$$F = -\kappa \sin\psi d\psi \wedge d\varphi \quad (36)$$

The RR4-form is

$$C_4 = -\frac{1}{y^4} dt dx_1 dx_2 dx_3 + 4\alpha_4 \quad (37)$$

where α_4 is a 4-form satisfying $d\alpha_4 = \text{volume of } S^5$. The induced metric becomes

$$ds_{D5}^2 = -\left(\frac{h}{y^2} - \omega^2 \sin^2\theta\right) dt^2 + \left(\frac{y^2/h + x_3^2}{y^2} + \theta'^2 + g'^2 \sin^2\theta\right) d\sigma^2 + 2\omega g' \sin^2\theta dt d\sigma \\ + \frac{dx_1^2 + dx_2^2}{y^2} y^2 + \cos^2\theta (d\psi^2 + \sin^2\psi d\varphi^2)$$

The sum of the induced metric and the gauge flux is

$$G_{D5} + F = \left[\begin{array}{cc} -\frac{1}{y^2} (h - \omega^2 y^2 \sin^2\theta) & \omega g' \sin^2\theta \\ \omega g' \sin^2\theta & \frac{1}{y^2} \left(\frac{y^2}{h} + x_3^2 + y^2 \theta'^2 \right) + g'^2 \sin^2\theta \end{array} \right] \frac{1}{y^2} \quad (38) \\ \left[\begin{array}{cc} \frac{1}{y^2} & \\ & \frac{1}{y^2} \end{array} \right] \left[\begin{array}{cc} \cos^2\theta & F' \\ -F' & \cos^2\theta \sin^2\psi \end{array} \right]$$

Then the DBI action is

$$S_{DBI} / T_5 = -2\pi TV \int dy d\psi \frac{\sqrt{\cos^4\theta \sin^2\psi + F'^2}}{y^4} L(\sigma) \quad (39)$$

The WZ term is

$$S_{WZ} / T_5 = -2\pi TV \int dy d\psi \frac{F'(\psi) x_3'}{y^4} \quad (40)$$

Summing them the action is

$$S_{D5} / T_5 = -2\pi TV \int \frac{dy d\psi}{y^4} \left[\sqrt{\cos^4\theta \sin^2\psi + F'^2} L(y) + x_3' F' \right] \quad (41)$$

We can determine the gauge flux in the same way as found in eq.(14). Substituting the gauge flux (36), the action is

$$S_{D5} / T_5 = -2\pi TV \int \frac{dy}{y^4} \left(\cos^4\theta + \kappa^2 L(\sigma) - \kappa x_3' \right) \quad (42)$$

$$L(\sigma) = \sqrt{\left(\frac{y'^2}{h} + x_3'^2 + y^2\theta'^2\right)\left(h - \omega^2 y^2 \sin^2\theta\right) + hy^2 g'^2 \sin^2\theta}$$

Equations of motion

For notational convenience, we define

$$\Theta := \sqrt{\cos^4\theta + \kappa^2}, S := \frac{y'^2}{h} + x_3'^2 + y^2\theta'^2, \Omega := h - \omega^2 y^2 \sin^2\theta \quad (43)$$

In this notation the action is

$$S_{D5} \sim \int d\sigma \frac{\Theta L - \kappa v}{y^4}, L = \sqrt{S\Omega + hg'^2 y^2 \sin^2\theta} \quad (44)$$

The equations of motion for y, θ, x_3 and g are

$$\begin{aligned} \frac{d}{d\sigma} \left(\frac{\Theta}{y^4} \frac{(y'/h)\Omega}{L} \right) + \frac{4}{y^5} (\Theta L - \kappa v) - \frac{\Theta}{y^4 L} (y\theta'^2 \Omega + y(hg'^2 - \omega S) \sin^2\theta) \\ - \frac{\Theta}{y^4 L} \left(-\frac{y'^2 \Omega}{h^2} + S + y^2 g'^2 \sin^2\theta \right) \partial h = 0 \end{aligned} \quad (45a)$$

$$\frac{d}{d\sigma} \left(\frac{\Theta}{y^4} \frac{\theta' y^2 \Omega}{L} \right) + \frac{2 \sin\theta \cos^3\theta}{\Theta} \frac{L}{y^4} - \frac{\Theta}{y^4} \frac{y^2 (hu^2 - \omega^2 S) \sin\theta \cos\theta}{L} = 0 \quad (45b)$$

$$\frac{d}{d\sigma} \left(\frac{\Theta}{y^4} \frac{x_3' \Omega}{L} - \frac{\kappa}{y^4} \right) = 0 \quad (45c)$$

$$\frac{d}{d\sigma} \left(\frac{\Theta}{y^4} \frac{hy^2 \sin^2\theta}{L} g' \right) = 0 \quad (45d)$$

We choose the gauge $L/\Theta = 1$, The equations of motion become

$$\frac{d}{d\sigma} \left(\frac{y'}{h} \frac{\Omega}{y^4} \right) + \frac{4}{y^5} (\Theta^2 - \kappa v) - \frac{1}{y^3} (\theta'^2 \Omega + y(hg'^2 - \omega S) \sin^2\theta) - \frac{1}{y^4} \left(-\frac{y'^2 \Omega}{h^2} + S + y^2 g'^2 \sin^2\theta \right) \partial h = 0 \quad (46a)$$

$$\frac{d}{d\sigma} \left(\theta' \frac{\Omega}{y^2} \right) + \frac{2 \sin\theta \cos^3\theta}{y^4} - \frac{(hu^2 - \omega^2 S) \sin\theta \cos\theta}{y^2} = 0 \quad (46b)$$

$$\frac{d}{d\sigma} \left(\frac{x_3' \Omega}{L} - \frac{\kappa}{y^4} \right) = 0 \quad (46c)$$

$$\frac{d}{d\sigma} \left(\frac{h \sin^2\theta}{y^2} g' \right) = 0 \quad (46d)$$

We define the following

$$A := -\frac{d}{d\sigma} \log \Omega = \frac{2\omega^2 y \sin\theta (y' \sin\theta + y\theta' \cos\theta) + 4r_H^4 y^3 y'}{\Omega} \quad (47a)$$

$$B := \frac{4h}{y\Omega} (\kappa v - \Theta^2), C := \frac{hg'^2 - \omega S}{\Omega}, D := -\frac{2\sin\theta \cos^3\theta}{y^2\Omega} \quad (47b)$$

$$E := \frac{h(S + y^2 g'^2 \sin^2\theta)}{\Omega} \partial h \quad (47c)$$

By solving the equations for the second derivative terms,

$$y'' = \frac{4y'^2}{y} + hy\theta'^2 + y'A + B + Chy \sin^2\theta + E \quad (48a)$$

$$\theta'' = \frac{2\theta'y'}{y} + \theta'A + D + C \sin\theta \cos\theta \quad (48b)$$

$$x_3'' = \frac{4x_3'y'}{y} - \frac{4\kappa y'}{y\Omega} + x_3'A \quad (48c)$$

$$g'' = -2\theta'g' \cot\theta + \frac{2y'g'}{y} - \frac{y'g'\partial h}{h} \quad (48d)$$

Boundary condition

The boundary condition is, since we fixed $\Theta/L = 1$, in $y \rightarrow 0$ limit,

$$\sqrt{\frac{\cos^4\theta_0 + \kappa^2}{\left(\frac{y'^2}{h} + x_3'^2 + y^2\theta'^2\right)(h - \omega^2 y^2 \sin^2\theta) + hy^2 g'^2 \sin^2\theta}} = \sqrt{\frac{\cos^4\theta_0 + \kappa^2}{y'^2 + x_3'^2}} = 1 \quad (49)$$

From the third equation of motion (48c)

$$\frac{4y'}{y\Omega} (x_3'(h - \omega^2 y^2 \sin^2\theta) - \kappa) \approx \frac{4y'}{y\Omega} (x_3' - \kappa) \quad (50)$$

From equations (49) and (50) we find the conditions,

$$x_3'(0) = \kappa, y' = \cos^2\theta_0 \quad (51)$$

We impose the Neumann condition $\theta'(0) = 0$ at the AdS boundary. The fourth equation (48d) gives

$$\frac{2g'}{y \sin\theta} (y' \sin\theta - y\theta' \cos\theta) \approx \frac{2g' \cos\theta_0}{y \sin\theta} (\cos\theta_0 \sin\theta_0 - y\theta'), g'(0) = 0 \quad (52)$$

Summarizing the appropriate boundary condition is

$$y(0) = 0, \theta(0) = \theta_0, x_3(0) \approx \kappa\sigma, g(0) = 0 \quad (53a)$$

$$y'(0) = \cos^2\theta', \theta'(0) = 0, x_3'(0) = \kappa, g'(0) = 0 \quad (53b)$$

Solution

We solve the equations of motion (48) under the boundary condition (53). The results of the numerical calculation are shown in the next seven figures. **FIG. 9** shows the behavior of x_3 as a function of y for different gauge fluxes. In this plot the angular velocity is

$\omega=0.5$ and the mass is $m=1$ as one can see the reflection point at the horizon $y=y_h$. For large values of κ the curve becomes smooth at the peak of y . **FIG. 10** shows the behavior of x_3 for different masses. For $m=0$ the relation of y and x_3 is linear. For non-zero mass the curve bends in the vicinity of the horizon.

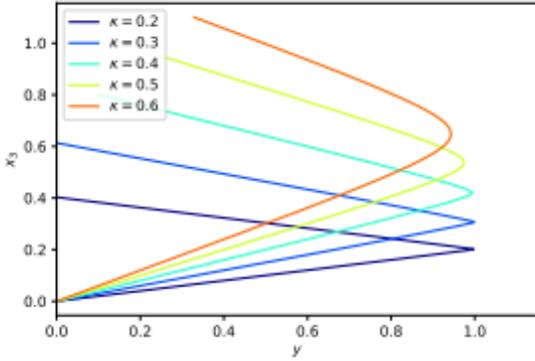


FIG. 9. κ dependence of x_3

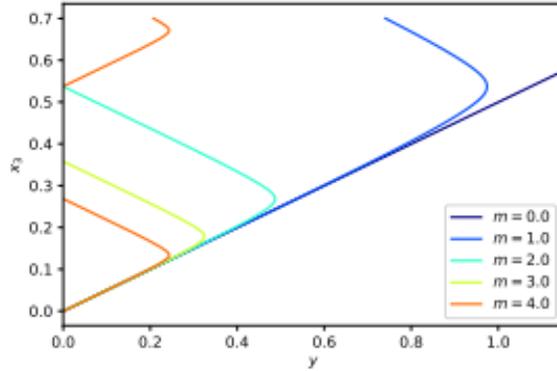


FIG.10. Mass dependence of x_3

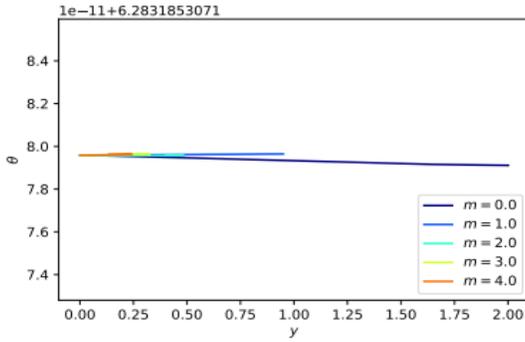


FIG.11. Mass dependence of θ

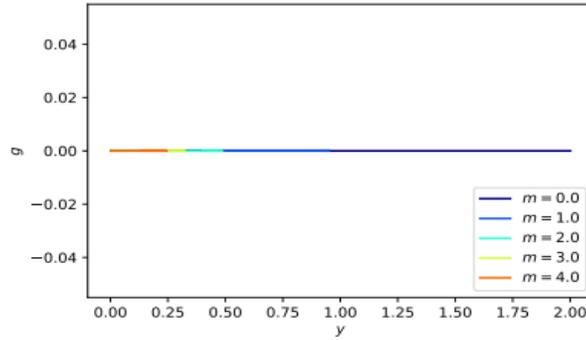


FIG.12. Mass dependence of g

Since the equations of motion are not explicitly dependent on x_3 , there are similar solutions which are parallel translated along x_3 direction. As we can see in **FIG.11** and **FIG.12**, θ and g do not grow as the same for the previous section. Since $\theta = 0$ is the solution, in the same way as before, there is no angular velocity dependence as shown in the next three figures (see **FIG.13**, **FIG.14** and **FIG.15**).

In $\kappa=0$ case we choose the gauge $\sigma=y$. Since identically $\theta = 0$, each factor become $L = \sqrt{1+x_3'^2}h$, $\Theta = \sqrt{1+\kappa^2}$ and $\Omega=h$. The equations (45) are simplified as

$$\frac{d}{dy} \left(x_3' \frac{h\sqrt{1+k^2}}{y^4 L} \right) = 0 \tag{54}$$

We can see that $x_3' = 0$ is the solution. This is the unique solution which can penetrate the horizon. In this case the action integrated over the Wheeler DeWitt patch is simply, $S_{D5} \sim \int_{y_h}^{\infty} dy / y^4$. The form of the D5-brane in (x_3, y) -plane is depicted in **FIG.16**.

Discussion

In the plots of the solution we found the behavior of a moving interface in subspace S^5 of $AdS_5 \times S^5$ black holes. In this paper, we found two things: The first is nontrans-parency of the horizon and the second is the independence of the angular velocity in S^5 subspace. Let us discuss them in detail. The D5-brane does not penetrate the horizon. In complexity-volume [14,31,32] and complexity-action [24,23] conjecture, we need to consider the surface or region.

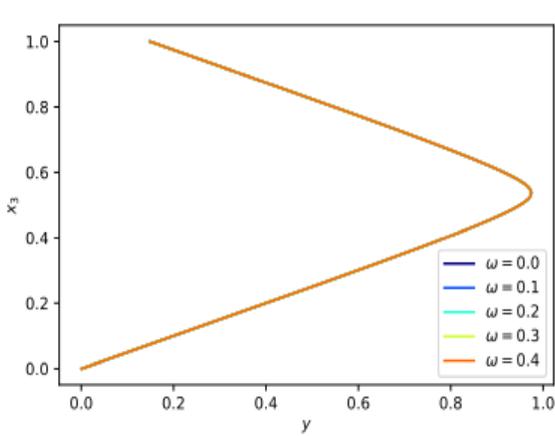


FIG. 13. ω dependence of x_3

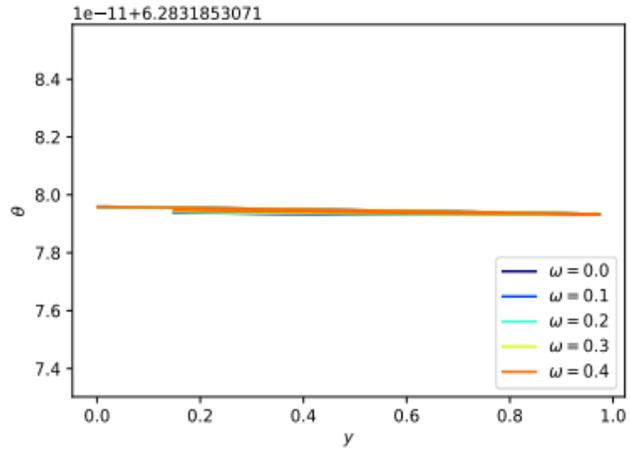


FIG.14. ω dependence of θ

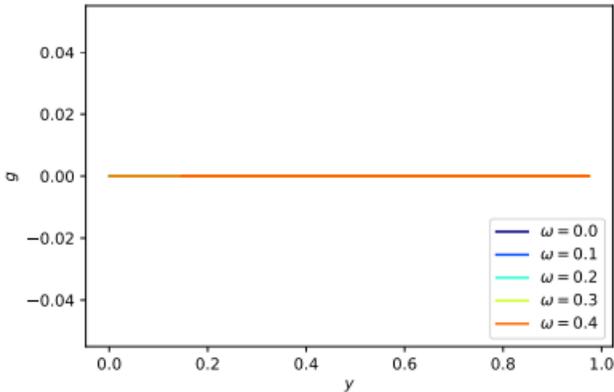


FIG.15. ω dependence of g

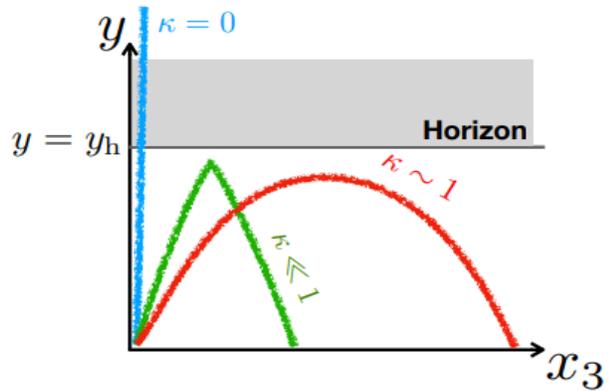


FIG. 16. Form of the D5-brane

which extends to the interior of the horizon. Especially the growth rate of the action is calculated from the integration inside the horizon. We want to find the effect of the non-local moving object such as [29] to black hole complexity. For this reason, we were looking for the case the probe brane penetrates the horizon. In [26,27] we found that the static probe brane does not penetrate the horizon. Then we tried to find a time dependent solution in this paper. According to the analysis, the probe D5-brane does not extend the horizon in the same way as the static case. We restricted the motion of the D5-brane in S^5 subspace. Then we found that the motion in S^5 part does not affect the embedding in AdS bulk. In order to find the probe brane which affects CA conjecture, we may need to consider a system which has the motion in AdS bulk. The second accomplishment is the discovery of the independence of the angular velocity in S^5 . That was due to the trivial solution for longitude coordinate $\theta=0$. This result says the behavior of the probe brane in the AdS_5 part is decoupled from the motion in the S^5 subspace. For this result, the interpretation in CFT side has another meaning because the action in AdS spacetime leads the drag force in the boundary theory [3]. In our case there are two interfaces on the boundary. The gauge theory on the each side of the interface has gauge group $SU(N)$ and $SU(N-k)$. In this work we focus on the motion restricted on a subspace of S^5 . For future work, we are interested in time dependent solutions moving on the AdS subspace.

We expect to find a solution which can exist in the horizon for non-zero gauge flux. This case gives non-trivial action which includes the gauge flux. This will be a generalization of (4.13) of [33] which includes the nonzero gauge flux. This gauge flux give us a useful way to compare the holographic quantities as stated in introduction. Then, if we find 12 the such time dependent solution, this give a good example to test CA conjecture for AdS black holes.

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