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The ruin problem of the renewal ri model with stochastic income

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ABSTRACT

In this paper, we consider the renewal risk model with stochastic income. In which, the premium process is modelled by a compound Poisson process and claim inter-arrival times are generalized Erlang (2) distributed. A system of integral equations for the discounted penalty function is derived. In the case of both premium and claim are exponential distributed closed form expression for Laplace transform of ruin probability is obtained, and a numerical answer of ruin probability as a example is given.

KEYWORDS

Stochastic income; Generalized erlang (2) distribution; Discounted penalty function; Laplace transform.

INTRODUCTION

The classical Cramer-Lundberg risk model commonly assume the premium as a linear function of time, that the premium aggregate at a constant rate. In practice, the charged premiums alter stochastically due to the fierce competition and continuous changing of business scope and performance of insurance companies. In this way, the literature [1] assumed the premium charging process in classical model to be a Compound Poisson process and obtained the upper bound on the probability of bankrupt. Discount penalty function, also known as Gerber-Shiu function, is used in literature [2] to do further research of the previously mentioned model, so as to obtain more actuarial variables.

Erlang distribution has been widely applied in queuing theory and control theory. Recently, Erlang distribution is also deployed in risk theory to depict the general renewal risk model, details can be found in literature [3-5]. Based on literature[1-2]and the assumption that the process of premium charging as a Compound Poisson process and that the claim occur interval satisfies generalized Erlang distribution, this article concluded that the discount penalty function of the model satisfies an integral equation. When the amount of premium and claim follow the exponential distribution, we obtain the analytical expression of the Laplace transformation of the bankrupt probability and give the numerical solution of bankrupt probability by an example.

MODEL INTRODUCTION

Assume a surplus process as follows：

$$U(t) = u + \sum_{i=1}^{N_1(t)} X_i - \sum_{i=1}^{N_2(t)} Y_i \qquad \textbf{(1)}$$

Where u is the initial surplus of the insurance company, $N_1(t)$ is a Poisson process with parameter λ, which is the number of premium collecting activity till moment t. If we assume the interval of premium charging to be $W_i, i = 1, 2, \cdots$,then $W_i \sim \exp(\lambda)$. X_i represents the amount of premium charged each time, with the distribution function $P(x)$ and the density function $p(x)$. $N_2(t)$ is a renewal process,expressing the claim arrival process. Assume the interval of claims to be $V_i = L_{i1} + L_{i2}, i = 1, 2, \cdots, L_{i1} \sim \exp(\lambda_1), L_{i2} \sim \exp(\lambda_2)$ and mutually independent, which infer that V_i satisfies generalized Erlang (2) distribution. Y_i expresses the amount of premium charged each time with the distribution function $F(x)$, and the relative density function $f(x)$. we hereby assume $N_1(t)$ and $N_2(t)$ to be mutually independent. In order to meet the stability condition, we have to assume $\lambda E(X) > \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} E(Y)$. Let the Gerber-Shiu function of the above surplus process Gerber-Shiu as follows,

$$\phi(u) = E\left[e^{-\delta T} \omega\left(U(T-), |U(T)|\right) I(T < \infty) \middle| U(0) = u\right], u \geq 0.$$

where T is the bankrupt time of the process, $\omega(x, y), x, y \geq 0$ is a bivariate function, $I(\cdot)$ is an indicator function. Since Gerber-Shiu function contains the actuarial variables, which includes bankrupt probability, the Laplace transform of bankrupt time and the distribution of surplus before ruin and the deficit at ruin, the major focus of this article is to search for the expression of $\phi(u)$.

In order to facilitate to give the expression of $\phi(u)$, this article then assumes a following auxiliary surplus process $\{U_1(t), t \geq 0\}$. This process is different from the surplus process $\{U(t), t \geq 0\}$ that the interval between the beginning and the initial claim in this process satisfies exponential distribution with parameter λ_2, while the subsequent intervals satisfy a generalized Erlang distribution (2) as $V_i\,(i = 2, 3, 4, \cdots)$. Except for the above mentioned differences, process $\{U_1(t), t \geq 0\}$ and process $\{U(t), t \geq 0\}$ share same assumptions. Assume the Gerber-Shiu function of the auxiliary surplus process $\{U_1(t), t \geq 0\}$ to be

$$\phi(u) = E\left[e^{-\delta T_1} \omega\left(U_1(T_1-), |U_1(T_1)|\right) I(T_1 < \infty) \middle| U_1(0) = u\right], u \geq 0.$$

Where T_1 is the time of ruin of this process.

THE INTEGRAL EQUATION OF $\phi(u)$

In this chapter, we applied the method in a similar literature[7] to surplus process (1) to discuss whether the cause of initial change of surplus is due to premium or claim.

Let $M = W_1 \wedge L_{11}$,then $\forall u \geq 0$,

$$\phi(u) = \int_0^\infty P\{M = t, M = L_{11}\} e^{-\delta t} \psi(u) dt$$

$$+\int_0^\infty P\{M = t, M = W_1\} e^{-\delta t} \left[\int_0^\infty \phi(u+x) dP(x)\right] dt, \quad (2)$$

If we let $M = W_1 \wedge L_{12}$, then $\forall u \geq 0$

$$\phi(u) = \int_0^\infty P\{Z = t, Z = L_{12}\} e^{-\delta t} \left[\int_0^u \phi(u-y) dF(y) + \int_u^\infty \omega(u, y-u) dF(y)\right] dt$$

$$+\int_0^\infty P\{Z = t, Z = W_1\} e^{-\delta t} \left[\int_0^\infty \phi(u+x) dP(x)\right] dt, \quad (3)$$

Here $f_{L_{11}}(y), y > 0$ is then density function of L_{11}.

Since

$$P\{M = t, M = L_{11}\} = P\{M = L_{11}\} \cdot P\{M = t | M = L_{11}\},$$

While

$$P\{M = L_{11}\} = P\{L_{11} < M_1\} = \int_0^\infty \left(1 - e^{-\lambda_1 x}\right) \lambda e^{-\lambda x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

And

$$P\{M = t | M = L_{11}\} = \frac{P\{L_{11} > t, W_1 > L_{11}\}}{P\{M = L_{11}\}} = \frac{\int_t^\infty P\{W > y | L_{11} = y\} f_{L_{11}}(y) dy}{P\{M = L_{11}\}} = e^{-(\lambda_1 + \lambda_2)t},$$

Hence

$$P\{M = t | M = L_{11}\} = (\lambda +) \lambda_1 e^{-(\lambda + \lambda_1)t}.$$

Similarly

$$P\{M = W_1\} = \frac{\lambda}{\lambda + \lambda_1}, P\{M = t | M = W_1\} = (\lambda + \lambda_1) e^{-(\lambda + \lambda_1)t},$$

So

$P\left\{M=t,M=L_{11}\right\}=\lambda_1 e^{-(\lambda+\lambda_1)t}$, $P\left\{M=t,M=W_1\right\}=\lambda e^{-(\lambda+\lambda_1)t}$.

Similarly we have

$P\left\{Z=t,Z=L_{12}\right\}=\lambda_2 e^{-(\lambda+\lambda_2)t}$, $P\left\{Z=t,Z=W_1\right\}=\lambda e^{-(\lambda+\lambda_2)t}$.

Let $W(u)=\int_u^\infty \omega(u,u-y)\,dF(y)$, and substitute it to formula（2）and（3）. In this way, we can obtain the follow theorem.

Theorem 2.1. The Gerber-Shiu functions of surplus process（1）and auxiliary surplus process $\left\{U_1(t),t\geq 0\right\}$ $\phi(u)$ and $\psi(u)$ satisfy the following integral equations,

$$\begin{cases}(\lambda+\lambda_1+\delta)\phi(u)=\lambda_1\psi(u)+\lambda\int_0^\infty \phi(u+x)\,dP(x)\\(\lambda+\lambda_1+\delta)\psi(u)=\lambda_2\int_0^u \phi(u-y)\,dF(y)+\lambda_2 W(u)+\lambda\int_u^\infty \psi(u+x)\,dP(x)\end{cases} \tag{4}$$

It is difficult to solve the above equations to obtain the analytic forms of $\phi(u)$ and $\psi(u)$, while we can find some relatively precise form of them in some certain conditions.

If we assume the distribution of the amount of charged premium X_i is $P(x)=1-e^{-ax}$, the distribution of the amount of claim each time Y_i is $F(x)=1-e^{-bx}$ and in the meanwhile $\omega(x,y)=1$, then $W(u)=\int_u^\infty dF(y)=e^{-bu}$. Let

$$M(u)\triangleq\int_0^\infty \phi(u+x)\,p(x)\,dx=ae^{au}\int_u^\infty \phi(y)e^{-ay}dy,$$

$$N(u)\triangleq\int_0^\infty \psi(u+x)\,p(x)\,dx=ae^{au}\int_u^\infty \psi(y)e^{-ay}dy,$$

$$H(u)\triangleq\int_0^\infty \phi(u-y)\,dF(y)=be^{-bu}\int_0^u \phi(t)e^{bt}dt.$$

Then then first equation of formula（4）is

$$(\lambda+\lambda_1+\delta)\phi(u)=\lambda_1\psi(u)+\lambda M(u) \tag{5}$$

Differentiating formula（5）with respect to u, we get

$$(\lambda+\lambda_1+\delta)\phi'(u)=\lambda_1\psi'(u)-\lambda a\phi(u)+\lambda aM(u) \tag{6}$$

Then substitute formula（6）in formula（5）, we get

$$C\phi'(u)=a(\lambda_1+\delta)\phi(u)+\lambda_1\psi'(u)-\lambda_1 a\phi(u) \tag{7}$$

Where $C=\lambda+\lambda_1+\delta$. The second equation in formula（4）can be rewritten as

$$D\psi(u)=\lambda_2 H(u)+\lambda_2 M(u)+\lambda N(u) \tag{8}$$

where $D=\lambda+\lambda_2+\delta$. Differentiating formula（8）with respect to u, we get

$$D\psi'(u) = \lambda_2 b\phi(u) - \lambda a\psi(u) - \lambda_2 b\left[H(u) + M(u)\right] + \lambda a^2 N(u) \quad (9)$$

Then differentiating formula (9) with respect to u, we get

$$D\psi''(u) = \lambda_2 b\phi'(u) - \lambda a\psi'(u) - \lambda_2 b^2\phi(u) - \lambda a^2\psi(u)$$

$$+\lambda_2 b^2\left[H(u) + M(u)\right] + \lambda a^2 N(u) \quad (10)$$

If we let

$$M = \lambda_2\left[H(u) + M(u)\right], N = \lambda N(u)$$

$$A = \lambda_2 b\phi(u) - \lambda a\psi(u)\ B = \lambda_2 b\phi'(u) - \lambda a\psi'(u)\ -\lambda_2 b^2\phi(u) - \lambda a^2\psi(u),$$

Then formula (8), (9), (10) can be written as

$$D\psi(u) = M + N \quad (11)$$

$$D\psi'(u) = A - bM + aN \quad (12)$$

$$D\psi''(u) = B + b^2 M + a^2 N \quad (13)$$

Calculate $(12)\times b + (13) - a\left[(11)\times b + (12)\right]$, we get

$$D\psi''(u) + E\psi'(u) + G(u) = \lambda_2 b\phi'(u) - \lambda ab\phi(u) \quad (14)$$

where $E = D(b-a) + \lambda a$, $G = -(\lambda_2 + \delta)ab$. Perform Laplace transform at both sides of formula （7） and（14） with the functions, we get

$$\begin{cases} \left[Cs - a(\lambda_1 + \delta)\right]\hat{\phi}(u) - c\phi(0) = (\lambda_1 s - \lambda_1 a)\hat{\psi}(u) - \lambda_1\psi(0) \\ (Ds^2 + Es + G)\hat{\psi}(u) - (Ds + E)\psi(0) - D\psi'(0) = (\lambda_2 bs - \lambda_2 ab)\hat{\phi}(u) - \lambda_2 b\phi(0) \end{cases} \quad (15)$$

Here $s \in C$. let $u = 0$ in formula（8） and（9）, we can get $D\psi'(0) = \lambda_2 b\phi(0) + \ a(\lambda_2 + \delta)\ \psi(0)$ after some calculation.

We substitute $-\lambda_2(b+a)$ in the second equation in formula（15）, we finally get

$$\begin{cases} \left[Cs - a(\lambda_1 + \delta)\right]\hat{\phi}(u) - c\phi(0) = (\lambda_1 s - \lambda_1 a)\hat{\psi}(u) - \lambda_1\psi(0) \\ (Ds^2 + Es + G)\hat{\psi}(u) - D(s+b)\psi(0) + \lambda_2(a+b) = \lambda_2(s-a)\hat{\phi}(u) \end{cases} \quad (16)$$

Obviously, if we know $\phi(0)$ and $\psi(0)$, then we can determine $\hat{\phi}(u)$ 和 $\hat{\psi}(u)$.

GENERALIZED LUNDBERG FUNDAMENTAL EQUATIONS

Let $T_0 = 0$, $T_k = \sum_{j=1}^{k} V_j$ express the k-th time when claim arrives, where $k = 1, 2, \cdots$, let

$$U_k = U(k) = u + \sum_{i=1}^{N_1(T_k)} X_i - \sum_{j=1}^{k} Y_j = u + \sum_{j=1}^{k}\left(\sum_{i=1}^{N(V_j)} X_{ij} - Y_j\right),$$

expressing the surplus process after the k-th claim. Here the latter equation expres $\{X_{ij}, i, j \geq 1\}$ are a sequence of i.i.d. random variables, and share the identical distribution with X_j. We can find a $s \in C$, such that process $\{e^{-\delta T_k + sU_k}\}$ is a martingale, that

$$E\left[e^{-\delta V_1 - sY_1 + s\sum_{i=1}^{N(V_1)} X_i}\right] = E\left[e^{-sY_1 + s\sum_{i=1}^{N(V_1)} X_i}\right] E\left[e^{-sY_1}\right] = 1 \quad \textbf{(17)}$$

While

$$E\left[e^{-\delta V_1 + s\sum_{i=1}^{N(V_1)} X_i}\right] = E\left[E\left[e^{-\delta V_1 + s\sum_{i=1}^{N(V_1)} X_i} \middle| V_1\right]\right] = E\left[e^{\left[\lambda(M_X(s)-1)-\delta\right]V_1}\right].$$

Here $M_X(s)$ is the moment generating function of X_i. $E\left[e^{-sY_1}\right] = \hat{q}(s)$ is the Laplace transform of Y_1. Since $F(x) = 1 - e^{-bx}$, $\hat{q}(s) =$

$\frac{b}{b+s}$, while $E\left[e^{-tV_1}\right] = \frac{\lambda_1\lambda_2}{(\lambda_1 - t)(\lambda_2 - t)}$.

If we let

$$\gamma(s) = \frac{\left[\lambda(M_X(s)-1) - (\delta + \lambda_1)\right]\left[\lambda(M_X(s)-1) - (\delta + \lambda_2)\right]}{\lambda_1\lambda_2},$$

The formula (17) equals to

$$\gamma(s) = \hat{q}(s), s \in C \quad \textbf{(18)}$$

The above equation is known as the generalized Lundberg basic equations.

Especially, when $P(x) = 1 - e^{-ax}$, $M_X(s) = \frac{a}{a-s}$. Thus formula (18) equals to

$$\left[Cs - a(\lambda_1 + \delta)\right]\left[Ds - a(\lambda_2 + \delta)\right](b + s) = b\lambda_1\lambda_2(s - a)^2 \quad \textbf{(19)}$$

Since $\gamma'(s) = \frac{2\lambda}{\lambda_1\lambda_2} M'_X(s)\left[\lambda(M_X(s)-1) - \frac{\lambda_1 + \lambda_2 + 2\delta}{2}\right]$, While $M'_X(s) \geq$

$E(X)>0$, then $\gamma'(s)=0$ has a positive root and is denoted as s_0, that $\lambda\left(M_X(s_0)-1\right)=\frac{\lambda_1+\lambda_2+2\delta}{2}$. When $s\in[0,s_0]$, we have $\gamma'(s)<0$. When $s\in[s_0,+\infty]$, we have $\gamma'(s)>0$ and $\gamma(s_0)=-\frac{(\lambda_1-\lambda_2)^2}{4\lambda_1\lambda_2}<0$. Since $\gamma(0)=\left(1+\frac{\delta}{\lambda_1}\right)\left(1+\frac{\delta}{\lambda_2}\right)>1=\hat{q}(0)$, $\gamma(s)=0$ has two different positive roots.

On the semicircle $\Gamma=\{s\in C:\mathbf{R}(s)>0,|s|=\rho,\rho>0\}$, when ρ is large enough, we have $|\gamma(s)|>|\hat{q}(s)|$. When $\mathbf{R}(s)=0$, $|\gamma(s)|>\left(1+\frac{\delta}{\lambda_1}\right)\left(1+\frac{\delta}{\lambda_2}\right)>1\geq|\hat{q}(s)|$. Therefore, on the line surrounded by the semicircle and the imaginary axis, we have $|\gamma(s)|>|\hat{q}(s)|$. According to Rouche's theorem, in interior of the contour, equation (18) has the same number of roots with $\gamma(s)=0$. That is equation $\gamma(s)=\hat{q}(s)$ has two positive real root, and is denoted as ρ_1,ρ_2, respectively.

CONCLUSION

Organizing formula (16) properly, we can have the expression of $\hat{\phi}(u)$ on $\phi(0)$ and $\psi(0)$.

$$\hat{\phi}(u)=\frac{\left[D\lambda_1(s+b)(s-a)-\lambda_1\left(Ds^2+Es+G\right)\right]\psi(0)}{\left[Cs-a(\lambda_1+\delta)\right]\left(Ds^2+Es+G\right)-\lambda_1\lambda_2 b(s-a)^2}$$

$$+\frac{C\left(Ds^2+Es+G\right)\phi(0)-\lambda_1\lambda_2(a+b)(s-a)}{\left[Cs-a(\lambda_1+\delta)\right]\left(Ds^2+Es+G\right)-\lambda_1\lambda_2 b(s-a)^2} \quad \textbf{(20)}$$

The denominator of the above formula is the equivalent form of the Lundberg equation mentioned in chapter 3. This formula is a cubic polynomial of s. if we let the denominator to be $D_3(s)$, then $D_3(s)=CD(s-\rho_1)(s-\rho_2)(s+R)$, then ρ_1,ρ_2 is the two positive real roots of Lundberg equation（18）. According to formula（20）, when $s=\rho_1,\rho_2$ and the denominator equals 0,since $\hat{\phi}(u)$ is finite, the numerator equals 0. Thus we have

$$\begin{cases}-a\lambda_1\lambda_2(\rho_1+b)\psi(0)+C\left[D\rho_1-a(\lambda_2+\delta)\right](\rho_1+b)\phi(0)-\lambda_1\lambda_2(a+b)(\rho_1-a)=0\\-a\lambda\lambda_1(\rho_2+b)\psi(0)+C\left[D\rho_2-a(\lambda_2+\delta)\right](\rho_2+b)\phi(0)-\lambda_1\lambda_2(a+b)(\rho_2-a)=0\end{cases} \quad \textbf{(21)}$$

With the help of formula (21), $\phi(0)$ and $\psi(0)$ can be solved.

$$\begin{cases}\phi(0)=\frac{\lambda_1\lambda_2(a+b)^2}{CD(\rho_1+b)(\rho_2+b)}\\\psi(0)=\frac{D\rho_1\rho_2-aD(\rho_1+\rho_2)-abD+ab(\lambda_2+\delta)+a^2(\lambda_2+\delta)}{a\lambda_1\lambda_2 D(\rho_1+b)(\rho_2+b)}\end{cases} \quad \textbf{(22)}$$

Substituting the $\psi(0)$ of formula（22）into formula（20）, we obtains another expression of $\hat{\phi}(u)$,

$$\hat{\phi}(u)=\frac{\lambda_1\lambda_2(a+b)\left[(s+b)(\rho_2-a)-(s-a)(\rho_2+b)\right]}{CD(s-\rho_1)(s-\rho_2)(s+R)(\rho_2+b)}+\frac{s+b}{(s-\rho_1)(s+R)}\phi(0) \quad \textbf{(23)}$$

The above formula is the analytical expression of Gerber-Shiu function $\phi(u)$ of the surplus process (1), when $\omega(x,y)=1$ and the amount of the premium charged and the claim both satisfy the exponential distribution.

Example 1. If we let $\lambda=\frac{1}{2},\lambda_1=1,\lambda_2=\frac{1}{2},a=2,b=\frac{1}{2},\delta=0$,then the basic function is $(0.5+s)$ $=0.0625(s-2)^2$. The corresponding roots are $\rho_1=0,\rho_2=1.3494,R=0.0494$.

Using Matlab to perform Laplace inverse transform, we obtain

$$\phi(u)=\frac{25000}{27741}e^{-\frac{247}{5000}u}.$$

This formula is the numerical expression of the probability of bankruptcy. When the initial surplus has different values, the corresponding probability of bankruptcy is shown in the following table

TABLE1 : The numerical results of probability of bankruptcy

u	**0**	**10**	**20**	**30**	**40**	**70**
$\phi(u)$	0.9012	0.5499	0.3355	0.2047	0.1249	0.0284

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