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The Research on the solution of variational inequalities

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ABSTRACT

Firstly, this paper presents the origin and definition of variational inequalities, the existence and uniqueness of the solution; and then to the unilateral stability problems in the elastic plate theory as the background, to discuss second classes of four order variational inequality, reset equivalent and the equation boundary value problem. Finally, it provides the foundation for the four order variational inequality of the second kind is solved by boundary element method.

KEYWORDS

Variational inequalities; The uniqueness of the solution; Second classes of four order variational inequality; The regularization method.

INTRODUCTION

Variational inequality is also known as the variational equation, originating in mathematical physics and nonlinear programming problems, which is a very important research field in Applied Mathematics; The model of variation inequality is partial differential equation with proper boundary value conditions and initial conditions, and it is a important branch of differential equation for it exist in equations.

Example 1

The variation principle is an important part of mathematical physics; take the stable of elastic film for example. Suppose the film in the region Ω of plane xy, the boundary is unchangble, and set elastic modulus and horizontal density is 1. The plan has an elastic deformation for vertical force exerted $f(x,y)$, and vertical displacement $u(x,y)$. In order to making potential reaching minimum by the principle of minimum potential. The problem is find the minimum of $u(x,y)$: solving $u \in H_0^1(\Omega)$, let

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v) \tag{1}$$

Among them

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y - \int_\Omega f\, u \, \mathrm{d}x\mathrm{d}y$$

In the other hands, by the principle of virtual work, the vertical displacement $u(x,y)$ meet virtual work equation

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \int_\Omega f\, u \, \mathrm{d}x\mathrm{d}y, \quad \forall v \in H_0^1(\Omega) \tag{2}$$

It can be proved easily by math that (1)equal to (2), and by calculus of variations, the euler equations of (1) is

$$\begin{cases} -\Delta u = f, & to\Omega \\ u = 0, & to\partial\Omega \end{cases} \tag{3}$$

Allow for the example in the application of variational principle for the entire set of functions linear space $H_0^1(\Omega)$, resulting in the equation in the form of (2) and (3). If allowed to the set of function is not the entire linear space but convex subset of this space, lead to the form of inequality problem that variational inequality. Therefore, variational inequalities variational principle is an important promotion.

Example 2

Suppose an elastic plate is placed on a rigid object, the plate surface Ω load $f(x,y)$, the rigid object surface is $\varphi(x,y)$, the plate boundary is $\partial\Omega$,

$$u = \frac{\partial u}{\partial v} = 0, \qquad to\partial\Omega$$

So, the allowable defection set is

$$K_4 = \{v \in H_0^2(\Omega) : v \geq \varphi, to\Omega\}$$

The deflection $u(x,y)$ is a solution of the minimization problem: solving $u \in K_4$ let

$$J(u) = \min_{v \in K_4} J(v) \tag{4}$$

Among them

$$J(v) = \frac{1}{2} a(v,v) - f(v)$$

$$a(u,v) = \int_\Omega \left(\Delta u \Delta v - (1-\sigma) \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \right) \right) \mathrm{d}x\mathrm{d}y$$

$$f(v) = \int_\Omega f\, v \,\mathrm{d}x\mathrm{d}y$$

Type (4) is equal to a variational inequality: solving $u \in K_4$,let

$$a(u, v-u) \geq f(v-u), \qquad \forall v \in K_4 \tag{5}$$

Can prove that $u(x,y)$ is also a solution of the linear complementarily problem, $u \in K_4$,and

$$\begin{cases} u \geq \varphi, \quad \Delta^2 u - f \geq 0, & in\Omega; \\ (u-\varphi)(\Delta^2 u - f) = 0, & in\Omega. \end{cases} \tag{6}$$

VARIATIONAL INEQUALITY AND ITS SOLUTION

Definition

Definition 1 The definition about the first class of elliptic variational inequalities is solving $u \in K$,let

$$a(u, v-u) \geq L(v-u), \qquad \forall v \in K \tag{7}$$

When $a(u,v)$ is symmetric, Type (7) is equal to a variational inequality:solving $u \in K$, let

$$J(u) = \min_{v \in K} J(v) \tag{8}$$

Among them

$$J(v) = \frac{1}{2} a(v,v) - L(v)$$

Definition 2 The definition about the second class of elliptic variational inequalities is solving $u \in V$,let

$$a(u, v-u) + j(v) - j(u) \geq L(v-u), \qquad \forall v \in V \tag{9}$$

The existence and uniqueness of the solution

Theorem 1 Suppose V is a Hilber space, and $K \subset V$ is a non empty closed convex set. Suppose $a(\cdot,\cdot): V \times V \to R$, which in space V , is a continuous, symmetric and elliptical bilinear form, ($L \in V'$), and $j: K \to R$ is a convex lower semi continuous function.

$$E(v)=\frac{1}{2}a(u,v)+j(v)-L(v)$$

The minimal value problem

$$u\in K, E(u)=\inf_{v\in K}E(v)$$

It has an exclusive solution. And that $u\in K$ is a minimal value problem solution, if and only if

$$u\in K,\quad a(u,v-u)+j(v)-j(u)\geq L(v-u),\quad \forall v\in K$$

The following gives a extension conclusion about theorem 1.

Suppose V is a real Hilbert space, it has the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We call the operator $A:V\to V$ is strongly monotone, if there is a constant $c_0>0$,

$$\left(A(u)-A(v),u-v\right)\geq c_0\left\|u-v\right\|^2,\quad \forall u,v\in V$$

We call the operator A is Linschitz continuous, if there is a constant $M>0$ meet

$$\left\|A(u)-A(v)\right\|\leq M\left\|u-v\right\|,\quad \forall u,v\in V$$

Theorem 2 Suppose V is a Hilber space, and $K\subset V$ is a non empty closed convex set. Suppose $A:V\to V$ strongly monotone and Linschitz continuous, $j:K\to R$ convex lower semi continuous. Then for any $f\in V$, elliptic variational inequalities

$$u\in K,\ \left(A(u),v-u\right)+j(v)-j(u)\geq\left(f,v-u\right),\quad \forall v\in K \tag{10}$$

Having an exclusive solution and the solution of u is Linschitz continuous dependence on f.

THE SECOND CLASS OF FOURTH ORDER VARIATIONAL INEQUALITIES AND ITS SOLUTION

We elasticity theory in flat background unilateral stability, discussion of the second class of fourth order variational inequalities, reset equivalent and the equation boundary value problem. Finally, it provides the foundation for the four order variational inequality of the second kind is solved by boundary element method.

Issues raised and symbols

Hypothesis Ω, which has a smooth boundary Γ, is a bounded open domain in R^2, meas $(\Gamma)>0$, we define $V=H^2(\Omega)\cap H_0^1(\Omega)$;

Among them

$$H^m(\Omega)=\left\{u\in L^2(\Omega);\partial^\alpha u\in L^2(\Omega),\left|\alpha\right|\leq m\right\}$$

$$H_0^1(\Omega)=\left\{u\in H^1(\Omega),u\big|_\Gamma=0\right\}$$

$$a(u,v)=\int_\Omega \Delta u\Delta v dx+\int_\Omega uvdx$$

$H^n(\Gamma)$, was defined as usually Sobolev space, n is a real number. We suppose $f\in L^2(\Omega)$ is a given function. Based on this definition, we then define

$$\langle f,v\rangle=\int_\Omega fvdx$$

$$j(v)=g\int_\Gamma\left|\frac{\partial v}{\partial n}\right|ds$$

Among them, $g\in L^\infty(\Gamma)$ is a given constant, and $g>0$ when it in Γ. We define the subspace Λ which belongs to $L^2(\Gamma)$ as follows

$$\Lambda=\left\{\mu(x)\middle|\ \mu(x)\in L^2(\Gamma),\left|\mu(x)\right|\le 1\quad a.e.\Gamma\perp\right\}$$

Consider the functional minimization problem

$$\begin{cases}solving\ u\in V,\quad made\\ J(u)\le J(v),\quad \forall v\in V\end{cases}\tag{11}$$

Among them

$$J(v)=\frac{1}{2}a(v,v)-\langle f,v\rangle+g\int_\Gamma\left|\frac{\partial v}{\partial n}\right|ds\tag{12}$$

Theorem 3 the functional minimization problem (11) is equivalent to solving the variational inequalities

$$\begin{cases}\qquad solving\ u\in V,\qquad made\\ a(u,v-u)+j(v)-j(u)\ge\langle f,v-u\rangle,\ \forall v\in V\end{cases}\tag{13}$$

Functional right side of inequality $j(\cdot)$ is non-differentiable

Proved: Suppose $u\in V$ is a minimum point in V which belongs to (11), the $\forall\ u\in V$ and $t\in[0,1]$ have

$$J(u)=\frac{1}{2}a(u,u)-\langle f,u\rangle+g\int_\Gamma\left|\frac{\partial u}{\partial n}\right|ds\le J\left(u+t(v-u)\right)$$

$$=\frac{1}{2}a\left(u+t(v-u),u+t(v-u)\right)-\langle f,u+t(v-u)\rangle+\ g\int_\Gamma\left|\frac{\partial u+t(v-u)}{\partial n}\right|ds$$

$$\le\langle f,u\rangle-t\langle f,v-u\rangle+(1-t)g\int_\Gamma\left|\frac{\partial u}{\partial n}\right|ds+tg\int_\Gamma\left|\frac{\partial v}{\partial n}\right|ds$$

Finishing

$$\frac{t}{2}a\left(v-u,v-u\right)+a\left(u,v-u\right)+g\int_{\Gamma}\left|\frac{\partial u}{\partial n}\right|ds-g\int_{\Gamma}\left|\frac{\partial v}{\partial n}\right|ds\geq\left\langle f,v-u\right\rangle$$

Let $t\to 0^{+}$ have to

$$a\left(u,v-u\right)+g\int_{\Gamma}\left|\frac{\partial u}{\partial n}\right|ds-g\int_{\Gamma}\left|\frac{\partial v}{\partial n}\right|ds\geq\left\langle f,v-u\right\rangle$$

It is the type (13).

On the contrary, $u\in V$ is a solution of the variational inequalities (13), because that

$$\frac{1}{2}a\left(v,v\right)-\frac{1}{2}a\left(u,u\right)=\frac{1}{2}\left[a\left(u+v-u,u+v-u\right)-a\left(u,u\right)\right]$$

$$=a\left(u,v-u\right)+\frac{1}{2}a\left(v-u,v-u\right)\geq a\left(u,v-v\right)$$

From type (13) we can obtain

$$\frac{1}{2}a\left(v,v\right)-\frac{1}{2}a\left(u,u\right)\geq a\left(u,v-u\right)\geq j\left(u\right)-j\left(v\right)+\left\langle f,v-u\right\rangle$$

So

$$\frac{1}{2}a\left(v,v\right)-\left\langle f,v\right\rangle+j\left(v\right)\geq\frac{1}{2}a\left(u,u\right)-\left\langle f,u\right\rangle+j\left(u\right)$$

That

$$J\left(v\right)\geq J\left(u\right)\quad\forall v\in V$$

So, the point $u\in V$ is a solution of functional minimization problem.

Friction problem and the corresponding reset equivalent and the equation boundary value problem.

Theorem 4 The solution of problem (13) could be represented by the following non-homogeneous reset equivalent and the equation boundary value problem:

Exist $\lambda\in\Lambda$ have to

$$\begin{cases}\Delta^{2}u+u=f & to\Omega\\ u=0,\Delta u=-g\lambda & in\Gamma\\ \lambda\frac{\partial u}{\partial n}=\left|\frac{\partial u}{\partial n}\right| & a.e.in\Gamma\end{cases}\qquad\textbf{(14)}$$

Proved: Let $u\in V$ Meet (13), By Green formula we can obtain

$$a(u,v-u)=\int_{\Omega}\Delta u\Delta(v-u)dx+\int_{\Omega}u(v-u)dx$$

$$=\int_{\Omega}\Delta^2 u(v-u)dx-\int_{\Gamma}\frac{\partial\Delta u}{\partial n}(v-u)ds+\int_{\Gamma}\Delta u\frac{\partial(v-u)}{\partial n}ds+\int_{\Omega}u(v-u)dx$$

$$\geq g\int_{\Gamma}\left|\frac{\partial u}{\partial n}\right|ds-g\int_{\Gamma}\left|\frac{\partial v}{\partial n}\right|ds+\int_{\Omega}f(v-u)dx$$

Thus

$$\int_{\Omega}\left(\Delta^2 u+u\right)(v-u)dx+\int_{\Gamma}\left[\Delta u\frac{\partial(v-u)}{\partial n}-\frac{\partial\Delta u}{\partial n}(v-u)\right]ds+$$

$$g\int_{\Gamma}\left[\left|\frac{\partial v}{\partial n}\right|-\left|\frac{\partial u}{\partial n}\right|\right]ds\geq\int_{\Omega}f(v-u)dx \qquad \textbf{(15)}$$

Taking $v=u+w$, among them

$$w\in H_0^2(\Omega)=\left\{w\in H^2(\Omega),w\big|_{\Gamma}=0\right\}$$

Obviously $v\in V$, and

$$(v-u)\big|_{\Gamma}=w\big|_{\Gamma}=0 \quad\Rightarrow\quad u\big|_{\Gamma}=v\big|_{\Gamma} \quad\Rightarrow\quad \frac{\partial u}{\partial v}\bigg|_{\Gamma}=\frac{\partial v}{\partial n}\bigg|_{\Gamma} \quad\Rightarrow\quad \left|\frac{\partial u}{\partial v}\right|\bigg|_{\Gamma}=\left|\frac{\partial v}{\partial n}\right|\bigg|_{\Gamma}$$

Thus

$$\int_{\Omega}\left(\Delta^2 u+u\right)wdx\geq\int_{\Omega}fwdx$$

The type is also established for $-w\in H_0^2(\Omega)$, so we can obtain

$$\int_{\Omega}\left(\Delta^2 u+u\right)(-w)dx\geq\int_{\Omega}f(-w)dx$$

That is

$$\int_{\Omega}\left(\Delta^2 u+u\right)wdx\leq\int_{\Omega}fwdx$$

So it has

$$\int_{\Omega}\left(\Delta^2 u+u\right)wdx=\int_{\Omega}fwdx \quad \forall w\in H_0^2(\Omega)$$

In the sense of generalized function, the variational method can be the basic lemma:

$$\Delta^2 u + u = f, \qquad to\Omega$$

Substituting into (15) where we have:

$$\int_\Gamma \left[\Delta u \frac{\partial(v-u)}{\partial n} - \frac{\partial \Delta u}{\partial n}(v-u) \right] ds + g\int_\Gamma \left[\left| \frac{\partial v}{\partial n} \right| - \left| \frac{\partial u}{\partial n} \right| \right] ds \geq 0$$

Finishing

$$\int_\Gamma \left[\Delta u \frac{\partial v}{\partial n} - \frac{\partial \Delta u}{\partial n} v + g\left| \frac{\partial v}{\partial n} \right| \right] ds - \int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| \right] ds \geq 0 \qquad \textbf{(16)}$$

Taking $v = ku$ in type (16), among them $k \geq 0$, therefore:

$$(k-1)\int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| \right] ds \geq 0 \quad \forall k \geq 0$$

When $0 \leq k < 1$, we can obtain

$$\int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| \right] ds \leq 0$$

When $k \geq 1$, we can obtian

$$\int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| \right] ds \geq 0$$

Thus

$$\int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| \right] ds = 0$$

That

$$\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left| \frac{\partial u}{\partial n} \right| = 0 \quad in\Gamma \qquad \textbf{(17)}$$

And because $u\big|_\Gamma = 0$, we can obtain

$$\Delta u \frac{\partial u}{\partial n} + g\left| \frac{\partial u}{\partial n} \right| = 0$$

That

$$\Delta u \frac{\partial u}{\partial n} = -g\left| \frac{\partial u}{\partial n} \right| \qquad \textbf{(18)}$$

We take the $\lambda^{-1} = \frac{\partial u}{\partial n} \cdot \left|\frac{\partial u}{\partial n}\right|^{-1}$, among them $\lambda \in \Lambda$.
So there exist $\lambda \in \Lambda$, meet

$$\lambda \frac{\partial u}{\partial n} = \left|\frac{\partial u}{\partial n}\right|$$

Substituting into (18), we can obtain

$$\Delta u = -g\lambda \ \text{ in}\,\Gamma$$

Conversely, if (14) holds.
Let $\Delta^2 u + u = f$ make inner product by $(v-u)$ in Ω, and then by using Green formula can be obtained:

$$a(u,v-u) + \int_\Gamma \frac{\partial \Delta u}{\partial n}(v-u)\,ds - \int_\Gamma \Delta u \frac{\partial(v-u)}{\partial n} ds - \int_\Omega f\,(v-u)dx = 0$$

Finishing

$$a(u,v-u) + \int_\Gamma \left[\frac{\partial \Delta u}{\partial n} v - \Delta u \frac{\partial v}{\partial n}\right] ds - \int_\Gamma \left[\frac{\partial \Delta u}{\partial n} u - \Delta u \frac{\partial u}{\partial n}\right] ds - \int_\Omega f\,(v-u)dx = 0 \qquad \textbf{(19)}$$

By (14) of known boundary conditions can be obtained

$$\int_\Gamma \left[\Delta u \frac{\partial u}{\partial n} - \frac{\partial \Delta u}{\partial n} u + g\left|\frac{\partial u}{\partial n}\right|\right] ds = \int_\Gamma \left[-g\lambda \frac{\partial u}{\partial n} + g\lambda \frac{\partial u}{\partial n}\right] ds = 0 \qquad \textbf{(20)}$$

While

$$\int_\Gamma \left[g\left|\frac{\partial v}{\partial n}\right| + \Delta u \frac{\partial v}{\partial n} - v\frac{\partial \Delta u}{\partial n}\right] ds \ge 0 \qquad \textbf{(21)}$$

By sum type (20) and type (21) can be obtained type (16).
Then by sum type (16) and type (19) we can obtain

$$a(u,v-u) + \int_\Gamma g\left|\frac{\partial v}{\partial n}\right| ds - \int_\Gamma g\left|\frac{\partial u}{\partial n}\right| ds - \int_\Omega f\,(v-u)\,dx \ge 0$$

So

$$a(u,v-u) + j(v) - j(u) \ge \langle f, v-u\rangle \qquad \forall v \in V$$

Thus the theorem can be proved.

The regularization method of the non differentiable term in second classes of four order variational inequality

In the solving process of this kind of Variational inequalities, one of the biggest questions is that it has existence No differential, this brings difficulties for construct and use of numerical methods. In this case, we use the regularization method to construct differentiable functional $j_\varepsilon(v) = g\int_\Gamma \Psi_\varepsilon\left(\frac{\partial v}{\partial n}\right)ds$ instead of $j(v) = g\int_\Gamma \left|\frac{\partial v}{\partial n}\right|ds$ No differential appreciatively, among of it.

$$\Psi_\varepsilon(\xi) = \int_0^\xi \varphi(t)dt = \begin{cases} g\xi - \frac{1}{2}\varepsilon g^2, & \xi \ge \varepsilon g \\ \frac{\xi^2}{2\varepsilon}, & |\xi| < \varepsilon g \\ -g\xi - \frac{1}{2}\varepsilon g^2, & \xi \le -\varepsilon g \end{cases} \tag{22}$$

Here

$$\varphi(t) = \begin{cases} g, & \xi \ge \varepsilon g \\ \frac{t}{\varepsilon}, & |\xi| \le \varepsilon g \\ -g, & \xi \le -\varepsilon g \end{cases} \tag{23}$$

Obviously

$$\lim_{\varepsilon \to 0} \Psi_\varepsilon(v) = g|v|$$

Accordingly, the variational inequality (13) can be approximated the following formula

$$\begin{cases} \quad\quad solving\ u \in V, \quad\quad made \\ a(u, v-u) + j_z(v) - j_z(u) \ge \langle f, v-u \rangle, \ \forall v \in V \end{cases} \tag{24}$$

Easy inspect Functional $j_\varepsilon(v)$ is convex, differentiable and $\lim_{\varepsilon \to 0} j_\varepsilon(v) = j(v)$

Theorem 5 Variational inequality (24) has a unique solution.

Next, our solution gives variational inequalities (24) and he solution of Variational inequality (13), their relation Satisfy.

Theorem 6 Let A and B is the solution of problem (13) (24), when $\varepsilon \to 0$, u_ε converges strongly to u.

Proof By the assumption, the problem (24) is

$$a(u_\varepsilon, v-u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \ge \langle f, v-u_\varepsilon \rangle \quad \forall v \in V$$

Let $v = u$, we have

$$a(u_\varepsilon, u-u_\varepsilon) + j_\varepsilon(u) - j_\varepsilon(u_\varepsilon) \ge \langle f, u-u_\varepsilon \rangle \tag{25}$$

In the formula (13), let $v = u_\varepsilon$,we have

$$a(u,u_\varepsilon - u) + j(u_\varepsilon) - j(u) \geq \langle f, u_\varepsilon - u \rangle \quad \textbf{(26)}$$

The (25) and (26) obtained by adding

$$a(u_\varepsilon - u, u_\varepsilon - u) \leq j_\varepsilon(u) - j(u) + j(u_\varepsilon) - j_\varepsilon(u_\varepsilon) \quad \textbf{(27)}$$

From (22), we obtain

$$0 \leq g|\xi| - \Psi_\varepsilon(\xi) \leq \frac{1}{2}\varepsilon g^2 \qquad \forall \xi \in R$$

Defined by A and B, we have

$$0 \leq j(v) - j_\varepsilon(v) \leq \frac{\varepsilon g^2}{2}\left(\int_\Gamma ds\right) \qquad \forall v \in V$$

Thus obtained by the equation (27)

$$a(u_\varepsilon - u, u_\varepsilon - u) \leq \frac{\varepsilon g^2}{2}\left(\int_\Gamma ds\right)$$

So, when $\varepsilon \to 0$, $a(u_\varepsilon - u, u_\varepsilon - u) \to 0$, use mandatory of $a(\cdot,\cdot)$, u_ε converges strongly to u. This proof completes.

Consider the following variational inequality (24) is equivalent form. Take $v = u \pm tw$, $t > 0$ $w \in V$ in (24), we have

$$\pm a(u,w) + g\int_\Gamma \frac{\Psi_\varepsilon\left(\frac{\partial u}{\partial n} \pm t\frac{\partial w}{\partial n}\right) - \Psi_\varepsilon\left(\frac{\partial u}{\partial n}\right)}{t} ds \geq \pm\langle f, w \rangle$$

Let $t \to 0^+$, attention to

$$\lim_{t \to 0^+} \frac{\Psi_\varepsilon\left(\frac{\partial u}{\partial n} \pm t\frac{\partial w}{\partial n}\right) - \Psi_\varepsilon\left(\frac{\partial u}{\partial n}\right)}{t} = \pm\Psi_\varepsilon^{'}\left(\frac{\partial u}{\partial n}\right)\frac{\partial w}{\partial n} = \pm\varphi\left(\frac{\partial u}{\partial n}\right)\frac{\partial w}{\partial n}$$

Solution of the problem (24) is $u \in V$, it meet

$$a(u,v) + g\int_\Gamma \varphi\left(\frac{\partial u}{\partial n}\right)\frac{\partial u}{\partial n} ds = \langle f, v \rangle \qquad \forall v \in V \quad \textbf{(28)}$$

This has been an equivalent variational form (28) of problem (24).
So, we can take advantage of the boundary element method to solve variational problem (28), we can take advantage of the boundary element method to solve variational problem (28).

CONCLUSION

The first, Defines variational inequalities of elastomeric frictional contact problems obey Coulomb’s law, and give the existence and uniqueness of the solution. And then, this paper defines the second class of fourth order variational inequalities, reset equivalent and the equation boundary value problem about the friction problem of elastic plate, and simplified boundary value problems and the corresponding homogeneous variational problems. The class of variational inequalities in non-differentiable term use of regularization method, using differentiable function let this problem into equivalent variational equations, and then, using the boundary element method to solve the question. This article is intended to provide a way of thinking for the second and fourth order for solving variational inequalities. In the future research work, In this way we will be extended to solve higher-order variational inequalities.

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