ISSN : 0974 - 7435 Volume 10 Issue 9

2014

BioTechnology

An Indian Journal

FULL PAPER

BTAIJ, 10(9), 2014 [3154 - 3159]

The research of nonlinear degenerate wave equations and its application on high accurarcy analysis

Zhiyan Li*, Linghui Liu, Chunguang Zhao, Shutao Jiang

School of Mathematics and Physics, Handan College, Handan, (CHINA)

ABSTRACT

The convergence analysis of the bilinear finite element method to a class of non-linear degenerate wave equation on anisotropic meshes is considered in this paper. Moreover, the global superconvergence for semidiscrete scheme is proposed through interpolation instead of the Ritz Volterra projection of the exact solution.

KEYWORDS

Nonlinear degenerate wave equation; Anisotropic meshes supercon- vergence.

INTRODUCTION

It is wellknow that wave equations can arise from many physical process, and a lot of them are nonlinear and widely used. Such as the perturked Sine-Gordon equation which occur in quantum mechanics, a model in the elasto–plastic- microstructure which describe the longitudinal motion of an elasto-plastic bar and anti-plane shearing in the case of spaces dimension (N=1). Therefore,many mathematicians and physicists focus their attention to study the nonlinear wave equations, and there have been a lot of impressive literatures[1-3].

The superconvergence study of the finite element methods is one of the most active research subjects both in theoretical and in practical computations. However, it seems that there are few studies focusing on the high accuracy analysis of finite element methods for the nonlinear wave equations with dissipation, especially on the anisotropic meshes. As a matter of fact, in many cases, the regularity or quasi-uniform assumption (i.e., $h_K / \rho_K \le C, \forall K \in T_h$,where T_h is a family of triangulation of Ω, h_K, ρ_K are the diameter of K and the biggest circle contained in K, respectively, $h = \max_{K \in T_h} h_K$ and C is a positive number independent of K and h.) described in[4] are great dificiences in application of finite element methods. For example, the solution of some elliptic problems may have anisotropic behavior in parts of the defined domain. This means that the solution only varies significantly in certain directions. It is an obvious idea to reflect this anisotropy in the discretion by using anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. Besides, some problems may be defined in narrow domain, for example, in modeling a gap between rotter and stator in an electrical machine, if we employ the regular partition of the domain, the cost of calculation will be very high. Therefore, to employ anisotropic meshes with fewer degrees of freedom is a better choice to overcome these difficulties. However, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to 0$. In this case, the Bramble-Hilbert Lemma can not be used in the estimate of the interpolation error. We have to apply the anisotropic finite element method as in[5-7].

In this paper, we consider the convergence analysis of the bilinear finite element method to a kind of nonlinear degenerate wave equation on anisotropic meshes. The superclose and superconvergence properties for semidicrete scheme is obtained based on the anisotropic interpolation theorem proposed in[5-7], and the integral identities developed in[8-10] with the help of interpolation of solution to the problem cosidered, instead of referring to the Ritz Volterra projection of the exact solution, which makes the proof rather simpler than the previous studies.

MODEL PROBLEM AND ITS VARIATIONAL FORMULATION

Consider the following nonlinear wave equation with dissipation:

$$\begin{cases} \mathbf{u_{tt}} - \Delta \mathbf{u} + \mathbf{g}(\mathbf{u_t}) + \mathbf{f}(\mathbf{u}) = \mathbf{0} & (\mathbf{x},\mathbf{t}) \in \Omega \times [\mathbf{0},\mathbf{T}] \\ \mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{0} & (\mathbf{x},\mathbf{t}) \in \partial\Omega \times [\mathbf{0},\mathbf{T}] \\ \mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{u_0}(\mathbf{x}), \mathbf{u_t}(\mathbf{x},\mathbf{0}) = \mathbf{u_1}(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \quad \mathbf{(1)}$$

where Ω is an open bounded set in R2, with smooth boundary $\partial\Omega$. g, f, u_0 and u_1 are all known functions. For simplicity, we assume:

(i) g and f are Lipschitz continuous with respect to u with Lipschitz constant L;

(ii) $u \in C^2(\Omega \times [0,T])$ is a unique solution of (1).

We denote by $W^{s,r}(\Omega)$ the standard Sobolev space of s-differential functions in $L^r(\Omega)$, its norm and seminorm by $\|\cdot\|_{s,r}$ and $|\cdot|_{s,r}$, and $H^2(\Omega) = W^{s,2}(\Omega)$, $\|\cdot\|_s = \|\cdot\|_{s,2}$, $0 \le S \le \infty, 1 \le r \le \infty$.

Throughout the paper C indicates a positive constant, possibly different accurrences, which is independent of the mesh parameters h, but may depend on u, g, f and T.

Then the weak form of (2.1) is to find $u(\cdot t):[0,T]\to H_0^1(\Omega)$, such that

$$\begin{cases}(\mathbf{u_{tt}},\mathbf{v})+(\nabla\mathbf{u},\nabla\mathbf{v})+(\mathbf{g}(\mathbf{u_t}),\mathbf{v})+(\mathbf{f},\mathbf{v})=\mathbf{0}, & \forall\mathbf{v}\in\mathbf{H_0^1}(\Omega),\\ \mathbf{u}(\mathbf{x},0)=\mathbf{u_0}(\mathbf{x}),\mathbf{u_t}(\mathbf{x},0)=\mathbf{u_1}(\mathbf{x}), & \mathbf{x}\in\Omega.\end{cases} \quad (2)$$

CONSTRUCTION OF THE FINITE ELEMENT

Let $\hat{K}=[-1,1]\times[-1,1]$ be the reference element with four vertices $\hat{a}_1=(-1,-1),\hat{a}_2=(1,-1),\hat{a}_3=(1,1)$ and $\hat{a}_4=(-1,1)$, and four edges $\hat{l}_1=\overline{\hat{a}_1\hat{a}_2}$, $\hat{l}_2=\overline{\hat{a}_2\hat{a}_3},\hat{l}_3=\overline{\hat{a}_3\hat{a}_4}$, and $\hat{l}_4=\overline{\hat{a}_4\hat{a}_1}$. The shape function space on $\hat{K}$ is defined as:

$$\hat{\boldsymbol{\Sigma}}=\{\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2,\hat{\mathbf{v}}_3,\hat{\mathbf{v}}_4\},\quad \hat{\mathbf{P}}=\mathbf{span}\{1,\xi,\eta,\xi\eta\} \quad (3)$$

where $\hat{v}_i=\hat{v}(a_i),i=1,2,3,4$.

It can be easily checked that the interpolations defined above are well-posed and the interpolation functions $\hat{I}\hat{v}$ can be expressed as:

$$\hat{I}\hat{v}=\frac{1}{4}(1-\xi)(1-\eta)\hat{v}_1+\frac{1}{4}(1+\xi)(1-\eta)\hat{v}_2+\frac{1}{4}(1+\xi)(1+\eta)\hat{v}_3+\frac{1}{4}(1-\xi)(1+\eta)\hat{v}_4.$$

It has been proved in[9] that the above interpolation operator has an anisotropic interpolation properties, i.e., for any $\alpha=(\alpha_1,\alpha_2),|\alpha|=1$,there holds

$$\left\|\hat{\mathbf{D}}^{\alpha}(\hat{\mathbf{v}}-\hat{\mathbf{I}}\hat{\mathbf{v}})\right\|_{0,\hat{\mathbf{K}}}\le \mathbf{C}\left|\hat{\mathbf{D}}^{\alpha}\hat{\mathbf{v}}\right|_{1,\hat{\mathbf{K}}}. \quad (4)$$

For the sake of convenience, let $\Omega\subset R^2$ be a convex polygon in x-y plane composed by a family of rectangular meshes $\Im_h$, which does not need to satisfy the regularity and quasi-uniform assumptions[4]. For any $K\in\Im_h$, denote the barycenter of element K by (xK; yK), the length of edges parallel to x-axis or y-axis by 2hx, 2hy, respectively, h_K = diam (K), $h=\max_{K\in\Im_h}h_K$.

Let $F_K:\hat{K}\to K$ be an affine mapping defined by

$$\begin{cases}\mathbf{x}=\mathbf{x_K}+\mathbf{h_x}\xi\\ \mathbf{y}=\mathbf{y_K}+\mathbf{h_y}\eta\end{cases} \quad (5)$$

Then the associated finite element space is

$$V_h=\left\{v\middle|\hat{v}=v|_K\circ F_K\in\hat{P},\forall K\in\Im_h,v|_{\partial\Omega}=0\right\},$$

Define the interpolation operator $I_h:H^2(\Omega)\to V_h$ as

$$I_h|_K=I_K,I_K:H^2(K)\to\hat{P}\circ F_K^{-1},I_Kv=(\hat{I}\hat{v})\circ F_K^{-1},\forall v\in^2(\Omega).$$

THE SEMI-DISCRETE SCHEME AND CONVERGENCE ANALYSIS

The approximation problem corresponding (4) reads as: Find $U(\cdot t):[0,T]\rightarrow V_h$, such that

$$\begin{cases}(U_{tt},v)+(\nabla U,\nabla v)+(g(U_t),v)+(f,v)=0, & \forall v\in V_h,\\ U(x,0)=Iu_0(x), U_t(x,0)=Iu_1(x), & x\in\Omega.\end{cases} \quad (6)$$

With the similar way to the proof in[10], we can get the semi-discrete (6) has a unique solution. From (4) and the Scaling techniques, we can easily get the following lemma.

Lemma 1 : If $u(\cdot,t),u_t(\cdot,t),u_{tt}(\cdot,t)\in H^2(\Omega),t\in[0,T]$, we have

$$\|(u-Iu)\|_0+h|u-Iu|_1\le Ch^2|u|_2, \quad (7)$$

$$\|(u-Iu)_{tt}\|_0+\|(u-Iu)_t\|_0+h|(u-Iu)_t|_1\le Ch^2(|u_t|_2+|u_{tt}|_2). \quad (8)$$

Theorem 1 : On the hypotheses of Lemma 1, and the assumptions are satisfied, there holds

$$\|(U-u)_t\|_0+h|U-u|_1\le Ch^2\left\{|u|_2+\left(\int_0^t\left(|u|_2^2+|u_t|_2^2+|u_{tt}|_2^2\right)d\tau\right)^{\frac{1}{2}}\right\}. \quad (9)$$

Proof : Let $U-u=U-Iu+Iu-u=\theta+\omega,\theta=U-Iu,\omega=Iu-u$ subtracts (2.2) from (6), for any $v\in V_h$, we get the error equation

$$(\theta_{tt},v)=(\nabla\theta,\nabla v)=(\omega_{tt},v)+(\nabla\omega,\nabla v)-(g(U_t)-g(u_t),v)-(f(U)-f(u),v). \quad (10)$$

Let $v=\theta_t$ in (4.5), then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\|\theta_{tt}\|_0^2+|\theta|_1^2\right\}&=(\omega_{tt},\theta_t)+\frac{d}{dt}(\nabla\omega,\nabla\theta)-(\nabla\omega_t,\nabla\theta)\\&-(g(U_t)-g(u_t),\theta_t)-(f(U)-f(u),\theta_t).\end{aligned} \quad (11)$$

Integrating (11) with respect to t, and noting that $\theta(0)=\theta_t(0)$, we have

$$\begin{aligned}\frac{1}{2}\left\{\|\theta_t\|_0^2+|\theta|_1^2\right\}&=\int_0^t(\omega_{tt},\theta_t)d\tau+(\nabla\omega,\nabla\theta)-\int_0^t(\nabla\omega,\nabla\theta)d\tau\\&+\int_0^t(g(U_t)-g(u_t),\theta_t)d\tau+\int_0^t(f(U)-f(u),\theta_t)d\tau\\&=\sum_{i=1}^{5}L_i\end{aligned} \quad (12)$$

Using Young inequality and Lemma 1, it follows that

$$L_1+L_2+L_3\le Ch^2\left(|u|_2^2+\int_0^t\left(|u_t|_2^2+|u_{tt}|_2^2\right)d\tau\right)+C\int_0^t\left(\|\theta_t\|_0^2+|\theta|_1^2\right)d\tau+\frac{1}{4}|\theta|_1^2 \quad (13)$$

By Lemma 1, assumption (i) and Cauchy inequality, yields

$$\begin{aligned} L_4 + L_5 &\le C\int_0^t \|U_t - u_t\|_0 \|\theta_t\|_0 d\tau + \int_0^t \|U - u\|_0 \|\theta_t\|_0 d\tau \\ &\le C\int_0^t \left(\|\theta_t\|_0^2 + \|\omega_t\|_0 \|\theta_t\|_0 + \|\omega\|_0 \|\theta_t\|_0 + \|\theta\|_0 \|\theta_t\|_0\right) d\tau \\ &\le Ch^2 \int_0^t \left(|u|_2^2 + |u_t|_2^2\right) d\tau + C\int_0^t \left(\|\theta_t\|_0^2 + \|\theta\|_0^2\right) d\tau. \end{aligned} \tag{14}$$

Combining (13) and (14), and applying Poincare' inequality, we see that

$$\|\theta_t\|_0^2 + |\theta|_1^2 \le Ch^2\left(|u|_2^2 + \int_0^t \left(|u|_2^2 + |u_t|_2^2 + |u_{tt}|_1^2\right) d\tau\right) + C\int_0^t \left(\|\theta_t\|_0^2 + |\theta|_1^2\right) d\tau . \tag{15}$$

From Gronwall's Lemma, there holds

$$\|\theta_t\|_0^2 + |\theta|_1^2 \le Ch^2\left(|u|_2^2 + \int_0^t \left(|u|_2^2 + |u_t|_2^2 + |u_{tt}|_1^2\right) d\tau\right). \tag{16}$$

Thus

$$\|\theta_t\|_0 + |\theta|_1^2 \le Ch\left(|u|_2^2 + \int_0^t \left(|u|_2^2 + |u_t|_2^2 + |u_{tt}|_1^2\right) d\tau\right)^{\frac{1}{2}}. \tag{17}$$

By the triangle inequality, the desired result then follows. The proof is completed.

THE SUPERCONVERGENCE ANALYSIS

Now, we will turn our attention to the superclose property. At first, we shall introduce the following important lemmas.

Lemma 2[11]: For any $v \in V_h$, $K \in \Im_h$, we have the following inequalities

$$\|v_{xy}\|_{0,K} \le Ch_x^{-1} |v_y|_{0,K}, \quad \|v_{xy}\|_{0,K} \le Ch_y^{-1} |v_x|_{0,K}. \tag{18}$$

Lemma 3: If $u(\cdot,t), u_t(\cdot,t) \in H^3(\Omega), t \in [0,T]$, for any $v \in V_h$; we have

$$(\nabla\omega, \nabla v)_K = O(h_K^2)|u|_{3,K} |v|_{1,K}, \tag{19}$$

$$(\nabla\omega_t, \nabla v)_K = O(h_K^2)|u_t|_{3,K} |v|_{1,K}. \tag{20}$$

Proof : Applying Lemma 2, with the similar way to the proof in[8], it follows that

$$(\omega_x, v_x) = O(h_y^2)|u|_{3,K} |v|_{1,K}, \tag{21}$$

$$(\omega_y, v_y) = O(h_x^2)|u|_{3,K} |v|_{1,K}. \tag{22}$$

Combining (21) and (22), we get (19). With the same techniques above we can derive (20). The proof is completed.

Applying the similar arguments as those in Theorem 1, and using the abave lemmas, we can easily obtain the following superclose property.

Theorem 2 : Assume u and U are the solution of (2) and (6), respectively. $u, u_t \in H^3(\Omega), u_{tt} \in H^2(\Omega)$,then

$$\|(\mathbf{U}-\mathbf{Iu})_t\|_0+|\mathbf{U}-\mathbf{Iu}|_1\le \mathbf{Ch}^2\left\{|\mathbf{u}|_3+\left(\int_0^t\left(|\mathbf{u}|_3^2+|\mathbf{u}_t|_3^2+|\mathbf{u}_{tt}|_2^2\right)d\tau\right)^{\frac{1}{2}}\right\} \quad (23)$$

In order to study the supercovergence results of the bilinear element for our problem, we construct the interpolation postprocessing operator I_{2h} as follows:

Combining four adjacent small elements K_1, K_2, K_3 and K_4 into one big element E the vertices of E are denote by $a_i(1\le i\le 9)$, the corresponding partition is denote by $\Im_{2h}$ =2h. For any $v\in H^3(\Omega)$, we define $I_{2h}v$ just the biquadratic Lagrange interpolation, i.e., for any $v\in H^3(\Omega), I_{2h}v\in Q_2$, and $I_{2h}v(a_i)=v(a_i), (1\le i\le 9)$.

From the anisotropic theorem proposed in[5, 6, 7], we know the interpolation operator I_{2h} satisfies the anisotropic property. Also, we can easily get the following lemma.

Lemma 5 : For any $v\in H^3(\Omega)$, the interpolation operator I_{2h} satisfy

$$\mathbf{I}_{2h}(\mathbf{Iv})\le \mathbf{I}_{2h}\mathbf{v}, \quad (24)$$

$$\|\mathbf{I}_{2h}\mathbf{v}\|_1\le \mathbf{C}|\mathbf{v}|_1, \quad \forall \mathbf{v}\in \mathbf{V}_h, \quad (25)$$

$$\|\mathbf{I}_{2h}\mathbf{v}-\mathbf{v}\|_1\le \mathbf{Ch}^2|\mathbf{v}|_3. \quad (26)$$

Theorem 3 : Under the assumption of Theorem 2, there holds

$$\|\mathbf{I}_{2h}\mathbf{U}-\mathbf{u}\|_1\le \mathbf{Ch}^2\left\{|\mathbf{u}|_3+\left(\int_0^t\left(|\mathbf{u}|_3^2+|\mathbf{u}_t|_3^2+|\mathbf{u}_{tt}|_2^2\right)d\tau\right)^{\frac{1}{2}}\right\} \quad (27)$$

Proof : Note that $I_{2h}U-u=I_{2h}U-I_{2h}u+I_{2h}u-u$, form Lemma 5 and Theorem 1, we get

$$\begin{aligned}\|\mathbf{I}_{2h}\mathbf{U}-\mathbf{u}\|_1&\le\|\mathbf{I}_{2h}\mathbf{U}-\mathbf{I}_{2h}(\mathbf{Iu})\|_1+\|\mathbf{I}_{2h}(\mathbf{Iu})-\mathbf{u}\|_1\\ &\le \mathbf{C}\left(\|\mathbf{U}-\mathbf{Iu}\|_1+\|\mathbf{I}_{2h}\mathbf{u}-\mathbf{u}\|_1\right)\\ &\le \mathbf{Ch}^2\left\{|\mathbf{u}|_3+\left(\int_0^t\left(|\mathbf{u}|_3^2+|\mathbf{u}_t|_3^2+|\mathbf{u}_{tt}|_2^2\right)d\tau\right)^{\frac{1}{2}}\right\}\end{aligned} \quad (28)$$

The proof is completed.

REFERENCES

[1] Liu Xiao-lan; China Sport Science and Technology, **29(13)**, 46-49 **(1984)**.
[2] Luo Yang-chun; Journal of Shanghai Physical Education Institute, **23(12)**, 46-47 **(1994)**.
[3] Wan Hua-zhe; Journal of Nanchang Junior College, **3**, 154-156 **(2010)**.
[4] Li Ke; Journal of Shenyang Sport University, **31(2)**, 111-113 **(2012)**.
[5] Zhang Shu-xue; Journal of Nanjing Institute of Physical Education, **31(2)**, 25-27 **(1995)**.
[6] Pan Li; Journal of nanjing institute of physical education (natural science), **19(1)**, 54-55 **(2004)**.
[7] Li Yu-he, Ling Wen-tao; Journal of Guangzhou Physical Education Institute, **17(3)**, 27-31 **(1997)**.
[8] Xu Guo-qin; Journal of Hebei Institute of Physical Education, **22(2)**, 70-72 **(2008)**.
[9] Chen Qing-hong; China Sport Science and Technology, **21(10)**, 63-65 **(1990)**.
[10] Tian Jun-ning; Journal of Nanjing Institute of Physical Education, **14(4)**, 149-150 **(2000)**.