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Super-efficiency infeasibility and zero data in DEA: An alternative approach

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ABSTRACT

The pitfall of infeasibility problem in most VRS radial super-efficiency models is a hot issue in data envelopment analysis (DEA) studies. Lee et al. (2012) [European Journal of Operational Research 216 (2012) 429–433] proposed a method to address the problem of infeasibility arises from zero input data in VRS super-efficiency DEA model. In this paper, we point out that their method can be replaced with an alternative approach and the main results are obtained identically from two methods. The proposed alternative method can also overcome the infeasibility problem caused by zero data in super-efficiency DEA models and have several advantages compared to Lee's model. Meanwhile, two numerical examples are utilized to illustrate validity and applicability of our alternative model.

KEYWORDS

Data envelopment analysis (DEA); Infeasibility; Super-efficiency; Evaluation.

INTRODUCTION

The classic radial super-efficiency model first developed by Andersen and Petersen[1] provides an effective means for enhancing the discrimination power of further recognizing efficient DMUs. It operates with the DMU under evaluation excluded from the reference set which is spanned by the remaining DMUs. However, the radial super-efficiency model may suffer from the pitfall of infeasibility problem under the condition of variable returns to scales (VRS).

In light of the problem, a numbers of articles have been tried to settle this problem. Lovell and Rouse[2] modify the conventional model by assigning a user-defined scaling factor to find a feasible solution for those efficient DMUs for which feasible solutions are unavailable in the VRS super-efficiency model. Chen[3] obtains the super-efficiency value with their corresponding virtual efficient projections for the DMUs whose observation data are infeasible. Cook et al[4]. develop a modified VRS super-efficiency model that yields optimal solutions and super-efficiency scores can characterize the extent of super-efficiency in both input and output level. Lee et al[5]. develop a two-stage process to address the VRS infeasibility issue. In the first stage, they test whether a VRS super-efficiency model is infeasible by investigating the existence of output surplus (input saving) when infeasibility occurs in the input-oriented (output-oriented) VRS super-efficiency model. In the second stage, they proposed a modified VRS super-efficiency model to yield a super-efficiency score that characterizes both the radial efficiency and input saving/output surplus. Chen and Liang[6] further prove that the two-stage process can be solved in a single linear program. However, when a DMU has zero data, these models may still be infeasible. Lee et al.[7] point out that zero output data will not lead to infeasibility of the output-oriented super-efficiency models mentioned above because the output side of the constraints can always be satisfied. Therefore, they only assume that some inputs are zero. Then, they proposed a revised model will be feasible when zero data exist in inputs.

The current paper points out that the method in Lee et al.[7] can be replaced with an alternative approach and the main results are obtained identically from two methods. The unit-invariant property and non-zero property can also be suitable to our method and several similarities and differences between two methods are compared.

The remainder of this paper is organized as follows. Section 2 looks back upon several radial super-efficiency models before and the problem of super-efficiency infeasibility. In Section 3 we show an alternative approach to overcome the infeasibility problem resulting from zero data. Meanwhile, two examples are used to demonstrate the usefulness of our approach. Section 4 compares our alternative approach with the method in Lee et al.[7] and shows several similarities and differences between two methods. In the end, some conclusions will follow in Section 5.

SUPER-EFFICIENCY MODELS

Suppose there are n DMUs, $\{\mathrm{DMU}_j\ (j=1,2,\cdots,n)\}$. Let $\{x_k, y_k\}$ denote the input and output vectors of the kth DMU. The ith input of the kth DMU is denoted as x_{ik} and the rth output of the kth DMU is denoted as y_{rk}.

The original input-oriented VRS super-efficiency model for efficient DMU_k can be expressed as:

$$\min \quad \theta$$

$$s.t. \quad \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} \le \theta x_{ik}, \quad i=1,\cdots,m$$

$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j y_{rj} \geq y_{rk}, \quad r=1,\cdots,s$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j = 1$$
$$\lambda_j \geq 0, \ j\neq k \tag{1}$$

Obviously, Model (1) is infeasible at least in the following two situations: one is that the DMU under consideration has the largest outputs. As pointed out in Lee et al.[5], when the outputs of the evaluated DMU is outside the production possibility set spanned by the outputs of the remaining DMUs, the infeasibility of input-oriented super-efficiency will occur, i.e., the second constrain condition cannot be met. The other case attributes to the fact that zero data exists in inputs of the evaluated DMU, i.e., the first constrain condition cannot be met.

Based on the works developed in Cook et al.[4] and Lee et al.[5], Chen et al.[6] proposed one model approach, where only one model (2) needs to be solved to obtain the super-efficiency score if model (1) is infeasible.

$$\min \quad \tau + M\times\sum_{r=1}^{s}\beta_r$$
$$s.t. \sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j x_{ij} \leq (1+\tau)x_{ik}, i=1,\cdots,m$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j y_{rj} \geq (1-\beta_r)y_{rk}, r=1,\cdots,s \tag{2}$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j = 1$$
$$\lambda_j \geq 0, \ j\neq k, \ \beta_r \geq 0.$$

where M is a user-defined large positive number. (In Cook et al.[4] and Lee et al.[5], M is set equal to 10^5.)

Next, Lee et al.[7] proposed the following model (3) to address the infeasibility resulting from that when some inputs are zero for some efficient DMUs. Here, $x_i^{\max} = \max_{1\leq k\leq n}\{x_{ik}\}$.

$$\min \quad \tau + M\times\left(\sum_{r=1}^{s}\beta_r + \sum_{i=1}^{m}t_i\right)$$
$$s.t. \sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j x_{ij} - t_i x_i^{\max} \leq (1+\tau)x_{ik}, i=1,\cdots,m$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j y_{rj} \geq (1-\beta_r)y_{rk}, r=1,\cdots,s; \tag{3}$$
$$\sum_{\substack{j=1\\j\neq k}}^{n}\lambda_j = 1$$
$$\lambda_j \geq 0, \ j\neq k, \ t_i \geq 0, \ \beta_r \geq 0$$

They deducted the term $t_i x_i^{\max}$ from the left side of the input constrains so that these constrains will not be violated when zero input occurs. Meanwhile, they explained two reasons for such conduct: the first is unit-invariant (unit-invariant property); the second is that $t_i x_i^{\max}$ will not be zero when x_{ik} is zero (non-zero property). And they proved that model (2) and model (3) yield the same results when data are positive.

THE ALTERNATIVE APPROACH

In fact, to meet the input constrains when zero input data exists, we can also substitute $x_i^{\max\{\bar{k}\}}$ for $x_i^{\max}$, where $x_i^{\max\{\bar{k}\}} = \max\limits_{j=1, j\neq k}^{n} \{x_{ij}\}$. Then, based on model (2), an alternative model can be similarly constructed as follows:

$$
\begin{aligned}
&\min \quad \tau + M \times \left(\sum_{r=1}^{s} \beta_r + \sum_{i=1}^{m} t_i \right) \\
&s.t. \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} - t_i x_i^{\max\{\bar{k}\}} \leq (1+\tau) x_{ik}, i = 1, \cdots, m \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} \geq (1-\beta_r) y_{rk}, r = 1, \cdots, s; \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0,\ t_i \geq 0,\ s_i \geq 0, \beta_r \geq 0,\ j \neq k
\end{aligned}
\tag{4}
$$

Note that as a result of $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1$, we have $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} - t_i x_i^{\max\{\bar{k}\}} \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_i^{\max\{\bar{k}\}} - t_i x_i^{\max\{\bar{k}\}} \leq (1-t_i) x_i^{\max\{\bar{k}\}}$. When t_i is large enough, even if $x_{ik} = 0$, model (4) is always feasible.

Model (1) is infeasible if and only if some $\beta_r^* > 0$ in model (4) (It can be readily deduced from Cook et al.[4], Lee et al.[5] and Chen et al.[6]). Further, the same conclusion can be easily proven that model (2) and model (4) yield the same results when data are positive. Comprehensively, model (4) can also handle the infeasibility occurring from those two situations mentioned above.

Model (3) and model (4) have a close relation that is revealed in the subsequent theorem.

Theorem

Model (3) and model (4) yield the same main results in optimality.

Proof

First note that $x_i^{\max}$ is equal to either $x_i^{\max\{\bar{k}\}}$ or x_{ik} for DMU_k $(k = 1, \cdots, n)$.

1) when $x_i^{\max} = x_i^{\max\{\bar{k}\}}$, model (3) is identical with model (4).

2) when $x_i^{\max} \neq x_i^{\max\{\bar{k}\}}$, i.e. $x_i^{\max} = x_{ik} \neq 0$, For a positive $x_{ik} \neq 0$, when $(1+\tau)$ is sufficiently large enough, t_i^* will be zero. The same thing happens when $x_i^{\max\{\bar{k}\}}$ exists.

So model (3) and model (4) yield the same main results in super-efficiency. □

This theorem indicates that when $x_i^{\max} = x_i^{\max\{\bar{k}\}}$ for the evaluated DMUs, i.e., when the current DMUs do not hold the largest *i*th input, model (3) and model (4) yield the same results in optimality. However, when $x_i^{\max} \neq x_i^{\max\{\bar{k}\}}$ for the evaluated DMUs, i.e., when the current DMUs hold the largest *i*th input, some difficult optimal values have been obtained by two models. Nevertheless, it is worth noting that some τ^*, t_i^* $(i = 1,\cdots,m)$ and β_r^* $(r = 1,\cdots,s)$ acquired differently from two models are too tiny to neglect, only leaving the same value of t_i^* and β_r^* that is large enough to consider. This is why the expression that "the same main results" is called. This result is related to the working blackbox in linear optimality, but it can be numerically testified (Two examples are listed in TABLE 1-6). We also check the alternative approach to the empirical case of illinois strip mines from Lee et al.[7], and the same main results are also achieved.

Further, the super-efficiency score can be defined as

$$\breve{\theta} = 1 + \tau_i^* + \hat{i} + \hat{o}$$

Similarly, then input savings index $\hat{i}$ and output savings index $\hat{o}$ can be defined in the following manner.

$$\hat{i} = \begin{cases} 0, & \text{if } I = \phi \\ \dfrac{\sum_{i\in I}\left(\dfrac{x_i^{\max\{\bar{k}\}} + t_i^* x_i^{\max\{\bar{k}\}}}{x_i^{\max\{\bar{k}\}}}\right)}{|I|} = \dfrac{\sum_{i\in I}\left(\dfrac{1+t_i^*}{1}\right)}{|I|}, & \text{if } I \neq \phi \end{cases} \qquad \hat{o} = \begin{cases} 0, & \text{if } R = \phi \\ \dfrac{\sum_{r\in R}\left(\dfrac{1}{1-\beta_r^*}\right)}{|R|}, & \text{if } R \neq \phi \end{cases}$$

where $R = \{r \mid \beta_r^* > 0\}$ and $I = \{i \mid t_i^* > 0\}$.

Because that $x_i^{\max}$ and $x_i^{\max\{\bar{k}\}}$ do not influence then input savings index $\hat{i}$, the identical super-efficiency score $\breve{\theta}$ is gained for each DMU based on model (3) and model (4) respectively.

COMPARING WITH LEE ET AL.

First, unit-invariant property and non-zero property can also be suitable for the use of deducting $t_i x_i^{\max\{\bar{k}\}}$ in our model (4) obviously. Then what we want to emphasize are these:

1) For those DMUs under assessment who have the largest data in some inputs, they get a quite different $x_i^{\max\{\bar{k}\}}$ compared to the other DMUs and the method of Lee et al.[7]. For instance, in TABLE 1, according to the method of Lee et al.[7], all DMUs have the same $x_1^{\max} = 3$, $x_2^{\max} = 4$. While following our approach, DMU_C with the largest $X_2 = 4$ has $x_2^{\max\{\bar{C}\}} = 3$, which is not equal to $x_2^{\max} = 4$, and DMU_E with the largest X_1 has $x_1^{\max\{\bar{E}\}} = 2$, which is not equal to $x_1^{\max} = 3$.

TABLE 1 : Numerical example 1 from Lee et al.[7]

DMU	X_1	X_2	Y_1	model (1)	model (2)
A	2	1	1	1	1
B	1	2	1	1.4	1.4
C	1	4	2	Infeasible	3
D	2	3	1	0.6	0.6
E	3	0	1	Infeasible	Infeasible

TABLE 2 : Results of numerical example 1 based on model (3)

DMU	τ^*	t_1^*	t_2^*	β_1^*	Input saving index	Output surplus index	Super-efficiency score
A	1.13E-11	2.45E-17	2.54E-17	2.22E-17	0	0	1
B	0.4	1.57E-21	2.39E-21	1.87E-21	0	0	1.4
C	7.13E-11	2.68E-16	2.24E-17	0.5	0	2	3
D	-0.4	5.49E-16	5.62E-16	5.56E-16	0	0	0.6
E	-0.33333	4.75E-17	0.25	6.42E-17	1.25	0	1.916667

TABLE 3 : Results of numerical example 1 based on model (4)

DMU	τ^*	t_1^*	t_2^*	β_1^*	Input saving index	Output surplus index	Super-efficiency score
A	1.13E-11	2.45E-17	2.54E-17	2.22E-17	0	0	1
B	0.4	1.57E-21	2.39E-21	1.87E-21	0	0	1.4
C	1.22E-10	3.30E-16	4.05E-17	0.5	0	2	3
D	-0.4	5.49E-16	5.62E-16	5.56E-16	0	0	0.6
E	-0.33333	4.70E-17	0.25	6.43E-17	1.25	0	1.916667

TABLE 4 : Numerical example 2

DMU	X_1	X_2	Y_1
A	2	1	1
B	1	2	1
C	1	4	2
D	10	0	1
E	0	8	1
F	10	10	1

TABLE 5 : Results of numerical example 2 based on model (3)

DMU	τ^*	t_1^*	t_2^*	β_1^*	Input saving index	Output surplus index	Super-efficiency score
A	0.53846	2.16E-20	3.25E-20	1.89E-20	0	0	1. 53846
B	0.4	6.37E-18	7.64E-18	7.10E-18	0	0	1.4
C	-0.2	1.66E-14	1.10E-14	0.5	0	2	2.8
D	-0.8	1.38E-14	0.1	2.13E-14	1.1	0	1.3
E	-0.75	0.1	2.39E-17	2.20E-17	1.1	0	1.35
F	-0.85	4.19E-17	4.35E-17	4.26E-17	0	0	0.15

TABLE 6 : Results of numerical example 2 based on model (4)

DMU	τ^*	t_1^*	t_2^*	β_1^*	Input saving index	Output surplus index	Super-efficiency score
A	0.53846	2.16E-20	3.25E-20	1.89E-20	0	0	1. 53846
B	0.4	6.37E-18	7.64E-18	7.10E-18	0	0	1.4
C	-0.2	1.66E-14	1.10E-14	0.5	0	2	2.8
D	-0.8	1.38E-14	0.1	2.13E-14	1.1	0	1.3
E	-0.75	0.1	2.39E-17	2.20E-17	1.1	0	1.35
F	-0.85	4.23E-17	4.37E-17	4.31E-17	0	0	0.15

2) The input constrain condition in our alternative approach is stronger via substituting $x_i^{\max\{\bar{k}\}}$ for $x_i^{\max}$. Due to $x_i^{\max\{\bar{k}\}} \le x_i^{\max}$, there are $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} - t_i x_i^{\max} \le \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} - t_i x_i^{\max\{\bar{k}\}} \le (1+\tau) x_{ik}$.

3) When the DMU$_k$ gets zero data in *i*th input, the corresponding t_i^* will be signally large (for example, $t_i^* > 10^{-10}$). At this moment, $x_i^{\max} = x_i^{\max\{\bar{k}\}} \neq 0$, so $t_i^* x_i^{\max\{\bar{k}\}} = t_i^* x_i^{\max}$, The term $t_i^* x_i^{\max\{\bar{k}\}}$ alsoreflects how far the DMU$_k$ is below the horizontal efficient boundary as $t_i^* x_i^{\max}$ means in Lee et al.[7].

4) As illustrated above, these two methods yield the same main results except that some intensity coefficients are different for DMUs with the largest input data in certain inputs. And these different intensity coefficients are too tiny to neglect.

5) By the way, the left side of both input and output constraints eliminate the inclusion of the inputs x_{ik} ($x_i^{\max}$ may be equal to x_{ik}) for the DMU$_k$ under evaluation by substituting $x_i^{\max\{\bar{k}\}}$ for $x_i^{\max}$, which formally accords with the original idea that the DMU under evaluation should be excluded from the reference set.

CONCLUSIONS

The conventional VRS radial super-efficiency model must suffer from the problem of infeasibility. A lot of works have been done in an effort to overcome this problem. To address this issue, the current paper provides an alternative approach for the method developed in Lee et al.[7]. We have shown several similarities and differences by comparing two methods and two examples are used to demonstrate that the same main results are obtained by these two methods.

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