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Strongly Prime Γ-Semigroup

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Abstract

The paper introduces the concepts of β-insulator and strongly prime-semigroups. Several characterizations of them are furnished.

Keywords: ***β*** **Insulator; Strongly prime; Right and left *α*–annihilator**

Introduction

The idea of general semigroups was developed by Anjaneyulu [1]. Saha defined Γ-semigroup as a generalization of semigroup as follows. Various kinds of Γ-semigroups have been widely studied by many authors [2-6].

In this paper we introduce and study the structure of β-insulator and strongly prime Γ-semigroups. In this paper many important results of strongly prime ideals in semigroups have been extended to strongly prime ideals in Γ-semigroups.

Prime and semiprime ideals of Γ-semigroups

Definition 1.1: A subset A of a Γ-semigroup S is said to be an m–system if $A=\phi$ or if $x, y \in A$ implies $<x>\Gamma<y> \cap A \neq \phi$

Definition 1.2: A subset A of a Γ-semigroup S is said to be an $\boldsymbol{n}$ – system if $A=\phi$ or if $x \in A$ implies $<x>\Gamma<x> \cap A \neq \phi$

Lemma 1.3: Let S be a Γ-semigroup. An ideal A in S is semiprime if and only if A^C is an $\boldsymbol{n}$ – system.

Proof: Suppose that A is a semiprime ideal and let $a \in A^c$. Then $a \notin A$. Since A is semiprime $<a>\Gamma<a> \not\subset A$. It implies that $<a>\Gamma<a> \cap A^C \neq \phi$ so that A^C is an $\boldsymbol{n}$ – system.

Conversely, suppose A^C is an ***n*** – system and let $a \notin A$. We shall prove that $<a> \Gamma <a> \not\subset A$. Since A^C is an ***n*** – system. $<a> \Gamma <a> \cap A^C \neq \phi$. Put $z \in <a> \Gamma <a> \cap A^C$. So that $z \in <a> \Gamma <a>$ and $z \notin A$. Hence $<a> \Gamma <a> \not\subset A$. Thus, A is a semiprime ideal.

Definition 1.4: For any ideal *Q* of a Γ-semigroup *S*, we define $n\ (Q)$ to the set of elements x such that every ***n*** – system containing *x* of *S* contains an element of *Q*.

Definition 1.5: An ideal *Q* in a Γ-semigroup *S* is said to be right primary if for any ideal *U* and *V*, $U\ \Gamma\ V$ implies $U \subseteq m$ (*Q*) or $V \subseteq Q$.

Theorem 1.6: Let *S* be a Γ-semigroup for any right primary ideal *P* in *S*, the following are equivalent

(i) *P* is a prime ideal.

(ii) $P=n$ (P).

(iii) *P* is a semiprime ideal.

Proof: (i)⇒(ii) Let *P* be a prime ideal then $P \subseteq n$ (P) is obvious. On the other hand, let $x \in n$ (P) and suppose that $x \notin P$. Since P is prime, P^C is an ***m***-system and $x \in P^C$. Then there exists an ***n***-system $N \subseteq P^C$ such that $x \in N$. But N is disjoint from P, therefore $x \notin n$ (P), which is a contradiction. Hence $x \in P$, so that n (P) $\subseteq P$

(ii) ⇒(iii) is obvious.

(iii) ⇒(i) Suppose that P is a semiprime ideal. We have to prove that P is a prime ideal. Let U and V be any ideal in S with $U\ \Gamma\ V \subseteq P$. Since P is primary, $U\ \Gamma\ V \subseteq P$ implies that $U \subseteq m\ (P)$ or $V \subseteq P$. Since P is a semiprime ideal, $P = m\ (P)$. Hence, $U \subseteq P$ or $V \subseteq P$. Thus P is a prime ideal in S.

Theorem 1.7: For any ideal P in S, P is prime if and only if P is primary and semiprime.

Proof: Suppose that P is a prime ideal. We have to prove that P is primary. Let U and V be any ideal in S such that $U\ \Gamma\ V \subseteq P$. Since P is a prime ideal, $U \subseteq n$ (P) or $V \subseteq P$ by theorem 2.6. Now our claim is that n (P) $\subseteq m\ (P)$. Let $x \in n$ (P) and S be any ***m*** **–** system containing *x*. Since is any ***m*** **–** system is an ***n*** **–** system, S is an ***n*** **–** system containing *x*. Since $x \in n$ (P), S meets P. Hence $x \in m$ (P) and therefore $U \subseteq n$ (P) or $V \subseteq P$ implies that $U \subseteq m$ (P) or $V \subseteq P$. Hence P is a primary ideal. Since every prime ideal is a semiprime ideal, *P* is semiprime and hence primary ideal.

Conversely, suppose that P is primary and semiprime ideal. By theorem 1.6, P is a prime ideal.

Strongly prime Γ-semigroups

Definition 2.1: Let S be a Γ-semigroup. Let S is said to be semiprime if 0 is a semiprime ideal. S is said to be prime if (0) is a prime ideal.

Definition 2.2: Let S be a Γ-semigroup. If *A* is a subset of S, we defined a right α-annihilator of *A* to be a right ideal $r_\alpha(A) = \{m \in S /\ A\alpha\ m=0\}$

Definition 2.3: Let S be a Γ-semigroup. If *A* is a subset of S, we defined a left α-annihilator of *A* to be a left ideal $l_\alpha(A) = \{m \in S / m\alpha\ A=0\}$

We adopt the symbol S^* to denote the nonzero element of S.

Definition 2.4: A right β-insulator for $a \in S^*$ is a finite subset of S, $M_\beta(a)$ such that $r_\alpha(\{a\beta c / c \in M_\beta(a)\}) = (0)$, for all $\alpha \in \Gamma$.

Definition 2.5: A left β-insulator for $a \in S^*$ is a finite subset of S, $M_\beta(a)$ such that $l_\alpha(\{c\beta a / c \in M_\beta(a)\}) = (0)$, for all $\alpha \in \Gamma$.

Definition 2.6: A Γ-semigroup S is sad to be a right strongly prime if for every $\beta \in \Gamma$, each non zero element of *S*, has a right β-insulator, that is for every $\beta \in \Gamma$ and $a \in S^*$, there is a finite subsest $M_\beta(a)$ such that for $b \in S$, $\{a\beta c\alpha b / c \in M_\beta(a)\}) = 0$, for all $\alpha \in \Gamma$ implies $b = 0$.

Definition 2.7: A Γ-semigroup *S* is sad to be a left strongly prime if for every $\beta \in \Gamma$, each non zero element of *S*, has a left β-insulator, that is for every $\beta \in \Gamma$ and $a \in S^*$, there is a finite subsest $M_\beta(a)$ such that for $b \in S$, $\{b\alpha c\beta a / c \in M_\beta(a)\} = 0$, for all $\alpha \in \Gamma$ implies $b = 0$.

Definition 2.8: A Γ-semigroup *S* is sad to be a left weakly semiprime Γ-semigroup if $[x, \Gamma] \neq 0$ for all $x \neq 0 \in S$.

Definition 2.9: A Γ-semigroup *S* is sad to be a right weakly semiprime Γ-semigroup if $[\Gamma, x] \neq 0$ for all $x \neq 0 \in S$.

Definition 2.10: A Γ-semigroup *S* is sad to be a weakly semiprime Γ-semigroup if it is both left and right weakly semiprime.

Theorem 2.11: Let *S* be a Γ-semigroup with D.C.C on annihilators then *S* is prime if *S* is strongly prime.

Proof: Suppose that *S* is right strongly prime. To prove *S* is prime, let $a, b \in S$ such that $a \neq 0$ and $b \neq 0$. Since *S* is right strongly prime, for every $\beta \in \Gamma$, there exists a right β-insulator $M_\beta(a)$ for *a*. Then $r_\alpha(\{a\beta c / c \in M_\beta(a)\}) = 0$, $\forall \alpha, \beta \in \Gamma$. Since $b \neq 0, b \notin r_\alpha(\{a\beta c / c \in M_\beta(a)\}), \forall \alpha, \beta \in \Gamma$, there exists $\alpha, \beta \in \Gamma$ such that $a\beta c\alpha b \neq 0$ where $c \in M_\beta(a)$. Hence *S* is prime.

Conversely, suppose that *S* is prime. We have to prove that *S* is right strongly prime. Let $s \in S^*$ and consider the collection of right α-annihilator ideals of the form $r_\alpha(\{s\beta n / n \in I\}), \forall \alpha, \beta \in \Gamma$ where *I* run over all finite subsets of *S* containing the identity. Since *S* satisfies the d.c.c. on right annihilators, choose a minimal element K. If $K \neq \{0\}$, we can find an element $a \in K$ such that $a \neq 0$. Since *S* is a prime Γ-semigroup, it follows from 2.6 theorem, that there exists $b \in S$, $s\gamma b\delta a \neq 0$, for $\gamma, \delta \in \Gamma$.

Let I' be a finite subset of *S* containing the identity and *b*. Since, $s\gamma b\delta a \neq 0,\ a \notin r_{\alpha}(\{s\beta n / n \in I'\})$, a contradiction. This forces that $K = \{0\}$. Thus, *s* has a right β-insulator $\forall \beta \in \Gamma$. Since $s \in S^*$ is arbitrary, every element of S^* has a right β-insulator $\forall \beta \in \Gamma$. Similarly, every element of S^* has a left β-insulator $\forall \beta \in \Gamma$. Hence *S* is a strongly prime Γ-semigroup.

Definition 2.12: Let *S* be a Γ-semigroup. A left ideal *I* of *S* is said to be essential if $I \cap J \neq 0$ for all nonzero left ideals *J* of *S*.

Definition 2.13: The singular ideal of a Γ-semigroup *S* is the ideal composed of elements whose right α-annihilator for each $\alpha \in \Gamma$ is an essential right ideal.

Theorem 2.14: If *S* is a strongly prime Γ-semigroup having no zero devisor, then singular ideal is zero.

Proof: Let *S* is a strongly prime Γ-semigroup and *A* be a singular ideal. Suppose that there exists an element $a \in A$ such that $a \neq 0$. Let $M_{\beta}(a)$ be a right β-insulator for *a*. Since *A* is an ideal, $a\beta b \in A, \forall b \in M_{\beta}(a)$. Now $r_{\alpha}(\{a\beta b\}) = \{x \in S / (a\beta b)\alpha x = 0\}$ implies that $a\beta b\alpha x = 0$, $\forall x \in r_{\alpha}(a\beta b),\ b \in M_{\beta}(a)$. Then $a\beta b\alpha$ $r_{\alpha}(\{a\beta b\}) = 0$. Hence, $a\beta b\alpha\ [\cap r_{\alpha}(\{a\beta b\})] = 0$. Since *A* is singular, $r_{\alpha}(\{a\beta b\})$ is essential for all $b \in M_{\beta}(a)$.

We know that the intersection of finitely many essential right ideals is nonzero. Since $M_{\beta}(a)$ is finite, $\bigcap_{b \in M_{\beta}(a)} r_{\alpha}(\{a\beta b\}) \neq 0$. Hence $r_{\alpha}(\{a\beta b / b \in M_{\beta}(a)\}) \neq 0$, which is a contradiction to the β-insulator $M_{\beta}(a)$. Consequently *A=0*.

Definition 2.15: Let S be a Γ-semigroup. Let us define a relation ρ on $SX\Gamma$ as follows: $(x,\alpha)\rho(y,\beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x,\alpha]$ denote the equivalence class containing $[x,\alpha]$. Let $L = \{[x,\alpha] : x \in S, \alpha \in \Gamma]\}$. Then L is a semigroup with respect to the multiplication defined by $[x,\alpha]\,[y,\beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup.

Theorem 2.16: If *S* is a right strongly prime Γ-semigroup, then the left operator Γ-semigroup $L(R)$ is right strongly prime Γ-semigroup.

Proof: Suppose that *S* is right strongly prime Γ-semigroup. To prove *L* is right strongly prime Γ-semigroup, it is enough to prove that every nonzero element in *L* has a right insulator. Let $\cup_i\ [x_i, \alpha_i] \neq 0\ \in$ *L*. Then there exists $x \in S$ such that $\cup_i\ [x_i, \alpha_i]x \neq 0$ that is $\cup_i\ x_i\alpha_i x \neq 0$. Since *S* is right strongly prime, for every $\beta \in \Gamma$, there exist an β-insulator for $\cup_i$ $x_i\alpha_i x$ say it $M_{\beta} = \{a_1, a_2, ..., a_n\}$. Then $r_{\alpha}(\{\cup_i\ \ x_i\alpha_i x\beta c / c \in M_{\beta}\}) = \{0\}, \forall \alpha, \beta \in \Gamma$. Hence for any $s \in S$, $(\cup_i$ $x_i\alpha_i x)\beta a_k \alpha s = 0\ \forall \alpha, \beta \in \Gamma$, $a_k \in M_{\beta} \Rightarrow s = 0$ (*). Now fix $\alpha, \beta \in \Gamma$, consider the collection $M'_{\beta} = \{[x\beta a_1, \alpha], [x\beta a_2, \alpha], ..., [x\beta a_n, \alpha]\}$. We shall prove that M'_{β} is an insulator for $[x_i, \alpha_i]$. It is enough to prove that $Ann(\{[x_i,\alpha_i]c'/c' \in M'_\beta\}) = \{0\}$. Let $\cup_i [x_i,\alpha_i] \in Ann(\{\cup_i [x_i,\alpha_i]c'/c' \in M'_\beta)\}=\{0\}$. Then $\cup_i [x_i,\alpha_i][x\beta a_k,\alpha] \cup_i [y_j,\beta_j] = 0, \forall k$. We claim that $\cup_i [y_j,\beta_j] = 0$. Now $\cup_i [x_i,\alpha_i][x\beta a_k,\alpha] \cup_i [y_j,\beta_j] = 0, \forall k$ implies that $\cup_i [y_j,\beta_j]\ s = 0, \forall s \in S$. Therefore $\cup_i [x_i,\alpha_i][x\beta a_k,\alpha] \cup_i [y_j,\beta_j](s)$.

i.e., $\cup_i [x_i\alpha_i x\beta a_k,\alpha] \cup_i y_j\beta_j s=0$.

i.e., $\cup_i [x_i\alpha_i x\beta a_k\alpha] \cup_i y_j\beta_j s=0$.

By (*), $\cup_i y_j\beta_j s=0$ i.e., $\cup_i [y_j,\beta_j]\ s = 0, \forall s \in S$. Hence, $\cup_i [y_j,\beta_j] = 0$. Since $\cup_i [x_i,\alpha_i] \neq \phi$ is arbitrary, every nonzero element in *L* has a right β-insulator. Similarly, if *S* is left strongly prime, then every non-zero element of *R* has a left β-insulator. Thus, *L* is right strongly prime, and *R* is a left strongly prime Γ-semigroup.

Theorem 2.17: A Γ-semigroup *S* is weakly semiprime then *S* is strongly prime and only if its left operator semigroup *L* is right strongly prime and its right operator semigroup *R* is left strongly prime.

Proof: Suppose that *L* is a right strongly prime Γ-semigroup. In order to prove that *S* is a strongly prime Γ-semigroup, we shall prove that for every $\beta \in \Gamma$, every non-zero element in *S* has a right β-insulator. Let $x \neq 0 \in S;\ \beta \in \Gamma$. Since *S* is a left weakly semiprime Γ-semigroup, $[x,\beta] \neq 0$. Since *L* is right strongly prime, there exists a right insulator $M([x,\beta]) = \bigcup_{j=1}^{n}[y_{jk},x_{jk}]/k = 1,2,......,s\}$ for $[x,\beta]$. Then $Ann(\{[x,\beta]c/c \in M([x,\Gamma])\}) = \{0\}$. Therefore, for any $U_i[z_l,\delta_l] \in L, [x,\beta]\bigcup_{j=1}^{n}[y_{jk},\beta_{jk}]\bigcup_i[z_l,\delta_l] = \{0\}$, for all $k = 1,2,......,s$ implies that $\bigcup_i[z_l,\delta_l] = 0.(**)$

Consider $M'_\beta = \{y_{jk}\beta_{jk}x/\ j = 1,2,.....,n; k = 1,2,.....,s\}$. We now claim that M'_β is a β-insulator for *x*. It is enough to prove that for each. Let $y \in r_\alpha(\{x\beta c/c \in M'_\beta\}), \forall \alpha \in \Gamma;$ then $(x\beta y_{jk}\beta_{jk}x)\alpha y = 0, \forall \alpha \in \Gamma$ and $k = 1,2,.....,s$. Therefore $[x\beta y_{jk}\beta_{jk}x\alpha y,\Gamma] = 0, \forall \alpha \in \Gamma$ adnd $k = 1,2,.....,s$. Hence $[x\beta y_{jk},\beta_{jk}][x\alpha y,\Gamma] = 0, \forall \alpha \in \Gamma$ and $k = 1,2,.....,s$ that is $[x,\beta][y_{jk},\beta_{jk}][x\alpha y,\Gamma] = 0, \forall \alpha \in \Gamma$ and $k = 1,2,.....,s$, so that $[x,\beta]\bigcup_{j=1}^{n}[y_{jk},\beta_{jk}]$ $[x\alpha y,\Gamma] = 0, \forall \alpha \in \Gamma$ and $k = 1,2,.....,s$. From (**), we have $[x\alpha y,\Gamma] = 0, \forall \alpha \in \Gamma$, so that $x\alpha y = 0, \forall \alpha \in \Gamma$. Since *S* is faithful $L \setminus R$ bimodule, we have *y=0*. Since $x \neq 0 \in S$ is arbitrary, for every $\beta \in \Gamma$, every non-zero element in *S* has a right β-insulator. Hence *S* is a right strongly prime Γ-semigroup. Similarly, if *R* is a left strongly prime Γ-semigroup then *S* is a left strongly prime Γ-semigroup. Converse part follows from Theorem 2.16.

Proposition 2.18: If *S* is strongly prime Γ-semigroup, then *S* is weakly semiprime Γ-semigroup.

Proof: Suppose that *S* is strongly prime Γ-semigroup. We shall prove that *S* is a weakly semiprime Γ-semigroup. Let $x \neq 0 \in S$. It is enough to prove that $[x,\Gamma] \neq 0$ and $[\Gamma,x] \neq 0$. Suppose that $[x,\Gamma] = 0$. Since *S* is a strongly prime Γ-semigroup, for every $\beta \in \Gamma$ there exists a finite subset $M_\beta(x)$ such that for $b \in S, \{x\beta c\alpha b/c \in M_\beta(x)\} = 0, \forall \alpha \in \Gamma$ implies that *b=0*. Now $x\beta c\alpha x = [x,\beta]c\alpha x = 0c\alpha x = 0, \forall \alpha,\beta \in \Gamma$. Hence *x=0*, a contradiction. Thus, *S* is a weakly semiprime Γ-semigroup.

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