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Stability and Hopf bifurcation in delayed Eco-epidem iological model for nonlinear incidence

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ABSTRACT

In this paper, a delayed eco-epidem iological model with nonlinear incidence is considered Firstly, the sufficient conditions of local stability of the positive equilibrium arc given by analyzing the linearized system characteristic value. Furthermore, conditions ensuring the existence of Hopf bifurcation are obtained. Then, based on center manifold and normal form, we get the formulas for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. At last, some numerical simulations are carried out to support the analytical results.

KEYWORDS

Eco-epidemiology;
Stability;
Bifurcations;
Nonlinear incidence;
Control.

INTRODUCTION

Since Kermark and Mekendrick in 1927, the first use of dynamic method to establish the mathematical model of infectious diseases (SIS), more and more mathematics, biology, workers are focused on research of epidemic dynamics. Especially in recent years, many authors of a large number of research involves the modeling and analysis of the appropriate mathematical model, to discuss the epidemic of infectious diseases and control[1-4].

In fact, on the one hand, a large number of epidemic dynamics model is involved in various aspects of the epidemic and the corresponding population, involves the epidemic of delay of stage-structure with the influence of the ecological model, and even some impact on the population of the second generation. On the other hand, there are some diseases spread between larvae only, such as cow rinderpest, seals of the masses of canine distemper, rabbit from the fluid tumor diseases and other diseases. Species in nature are compete with other species food resources, outside of habitats or prey on other species, so it is necessary in the population dynamics model on the basis of considering the influence of the disease, or is on the basis of the epidemic model considering the influence of the interaction between a population. We combine dynamics and population dynamics of infectious diseases to study is more accord with the actual situation of the ecological epidemic, the purpose of this research is to explore the mechanism of the epidemic, combined with their influence on the ecological environment, how to improve the ecological environment in the

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popularity of to control the disease, improve the quality of the survival of biological and population is of great significance to[5-23].

In the dynamic model of infectious disease process, in order to facilitate the study, the incidence of selected basic substantially traditional role of a simple substance ratio, i.e., Bilinear form. But the literature[4] described a simpler Bilinear function of the incidence model, general nonlinear dynamic model of the incidence of infectious diseases are likely to become more complex. Based on these reasons, the literature[5-7] considered the nonlinear dynamics model with general incidence of infectious diseases, and has been a series of rich results. Literature[7-8] analyzed the stability of the equilibrium point and discuss the bifurcation[9]. analyzed the local stability and global stability of the equilibrium point. [France] studied the nonlinear equations with Delay global asymptotic stability, and discuss the conditions and generate bifurcation of periodic solutions. Literature[11] consider a class of delay eco-epidemiological model to study its stability and bifurcations discussion, the conditions have been generated[12]. literature to consider a class has a two-stage structure two-delays epidemic model, get to know positive and bounded nature, through distribution analysis equation roots, two-delay parameters discussed the balance point of stability and existence of the local branch. The use of canonical form and the central flow pattern theory, the decision to branch direction and stability of the bifurcating periodic solutions implicit algorithm, using some numerical simulations to support the theoretical results obtained to say.

Under the foundation works of[11], by using of method in literature[12], for a class nonlinear function of birth rate to establish Biological and epidemic dynamics model, to effort study the stability and Hopf bifurcation problem. We consider Biological and epidemic dynamics model as follows.

$$\begin{cases} \dot{X} = -r_2 X^* X(t) - eX^* S(t), \\ \dot{S}(t) = kS^* e^{-d_1\tau} X(t-\tau) + \left(kX^* e^{-d_1\tau} - K\right) S(t-\tau) + \left(K - \left(ke^{-d_1\tau} X^* + \beta I^* \left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right)\right)\right) S(t) \\ \qquad - d_2 e^{-d_1\tau} I(t), \\ \dot{I}(t) = \beta f'\left(S^*\right) I^* S(t). \end{cases} \quad (1)$$

where $X(t)$ denotes the prey population, the prey in the absence of any predator grows in a Logistic way. $S(t)$ and $I(t)$ are susceptible and infected pests at time t, respectively. The parameter r_1 is the intrinsic rate of increase, is the bulk restrict factor, the parameter β denoted the transmission coefficient, τ is the rate for gestation(incubation) of the prey population in susceptibleÿ k_1 is transform factor, let $k = k_1 e$, where $0 < k_1 \le 1$, e is predation ratio, d_1 is the rate for natural death rate of the prey population in susceptible, d_2 is the rate for natural death rate of the prey population at infected pests. $f(S)$ is incidence, it satisfy $f(0)=0, f'(S)>0$, K is feedback control constants, it are positive constants.

Particularly when $K < 0$, said susceptible population pregnancy number is greater than the number of non pregnancy susceptible population; When $K > 0$, said the number of susceptible people pregnancy less than the number of pregnancy of susceptible population, obviously infected person will have a direct impact to the next generation of survival quality, the need for scientific prediction and effective control. In this paper, the model of corresponding model (1) than in the past[11] better and more academic and application value.

The initial condition of system (1) as follows :

$$X(\theta) = \phi_1(\theta), S(\theta) = \phi_2(\theta), I(\theta) = \phi_3(\theta), \theta \in [-\tau, 0], \phi_1(\theta), \phi_2(\theta), \phi_3(\theta) \ge 0 \quad \textbf{(2)}$$

This construction of this paper as followsÿwe will discusses these existence condition of locally stability and *Hopf* -bifurcation for positive equilibrium point in second section; By using of Literature[10] in three section, we introduce normal type methodsÿto give the direction of Hopf-bifurcation of system(1.1), and the stability of bifurcation for periodic solution and properties for compute formula etc. we will pay numerical simulation(experiments)will for the birth rate in not same case stating in four section, to support the theory and analyses that results for us.

BOUNDED AND POSITIVE OF SOLUTION

In the following we give out the results of this section, from ecological significance,

we assume $\tau_1 \le \tau$. Let $\Phi = (\psi, \phi_1, \phi_2)$,and system(1.1)satisfy initial value condition:

$$\psi, \phi_1, \phi_2, \in C\left([-\tau, 0], R_+^3\right),\ \psi(0) > 0, \phi_1(0) > 0, \phi_2(0) > 0 \quad \textbf{(3)}$$

where $R_+^3 = \{(X, S, I) : X \ge 0, S \ge 0, I \ge 0, i = 1, 2\}$.

Theorem 2.1 Assume that as $t \in [-\tau, 0]$, and $\Phi(t) > 0$, then the any solution of system (1) is strictly positive.

Proof. First, we show that as $t > 0$, $X(t) > 0$ and $I(t) > 0$. By the first equation of (1) and integral for the three equationÿwe get easy

$$X(t) = X(0)e^{\int_0^t (r_1 - r_2 X(s) - eS(s))ds} > 0 \quad I(t) = I(0)e^{\int_0^t \left(\beta e^{-d_1\tau} f(S(s))I(s) - d_2 I(s)\right)ds} > 0,\ t > 0.$$

Next, shows that when $t > 0$,that $S(t) > 0$,assume that exists t_0 ÿsuch that $S(t_0) = 0$ ÿand assumeis first value to suiteÿthat is let

$$t_0 = \inf\{t > 0 : S(t) = 0\} \quad \textbf{(4)}$$

So,

$$\dot{S}(t_0) = ke^{-d_1\tau} S(t_0 - \tau) X(t_0 - \tau) - \beta f(S(t_0)) I(t_0) - d_1 S(t_0) + K(S(t_0) - S(t_0 - \tau)) = S(t_0 - \tau)\left(ke^{-d_1\tau} X(t_0 - \tau) - K\right) > 0.$$

Hence, for sufficiently small $\varepsilon > 0$, $\dot{S}(t_0 - \varepsilon) > 0$. From the definition of t_0, $\dot{S}(t_0 - \varepsilon) \le 0$.

This is a contradiction. Therefore, we have $S(t) > 0$.

Theorem 2.2 Assume under $e \ge k$ and initial condition (3), the solution of system (1.1) is in lasting bound.

Proof. Definition $\xi(t) = (X(t), S(t), I(t))^t$, let $V(\xi(t)) = X(t) + S(t) + I(t)$, here $V(\xi(t)) : R_+^3 \to R_+$. From(1.1) we have

$$\dot{V}(t) = \dot{X}(t) + \dot{S}(t) + \dot{I}(t)$$
$$= (r_1)X(t) - (d_1 - K)S(t) - d_2 I(t) - r_2 X^2(t) - eX(t)S(t) - KS(t-\tau) + kS(t-\tau)x(t-\tau)$$
$$\le -rV(\xi(t)) + 2(r_1)X(t) - r_2 X^2(t) + (k - e)X(t)S(t)$$
$$\le -rV(\xi(t)) + 2(r_1)X(t) - r_2 X^2(t) + \frac{k-e}{2}\left(X^2(t) + S^2(t)\right)$$
$$= -rV(\xi(t)) + 2(r_1)X(t) - \left(r_2 + \frac{e-k}{2}\right)X^2(t) - \frac{(e-k)}{2}S^2(t)$$

Here, $r = \min\{r_1, d_1, d_2\}$, then there is $M > 0$, such that $\dot{V} + rV \le M$. From compare

Theorem, there is some a constant C, suites $V(t) \le (M/r) + Ce^{-rt}$, here

$V(t_0) = V(0) \le (M/r) + C$. Check the positive constant $M^* > (M/r)$ near the M/r

sufficiently, and define $\Omega = \{(X, S, I) \in R_+^3 : V(t) \le M^*\}$, therefore, Ω is in lastly bound region of system (1.1). That is existence a finite time T, such that if $t > T$,

$\xi(t) \in \Omega$, and $T = T(t_0, \xi(t_0)) = \max\left\{\ln\left(C/\left(M^* - M/r\right)\right)^{1/r}, 0\right\}$, here $\xi(t_0) = (X(0), S(0), I(0))$.

STABILITY AND HOPF-BIFURCATION

We assume the system:

$$\begin{cases} \dot{X} = -r_2 X^* X(t) - eX^* S(t), \\ \dot{S}(t) = kS^* e^{-d_1\tau} X(t-\tau) + \left(kX^* e^{-d_1\tau} - K\right) S(t-\tau) + \left(K - \left(ke^{-d_1\tau} X^* + \beta I^* \left(f'\left(S^*\right) - \dfrac{f(S^*)}{S^*}\right)\right)\right) S(t) \\ \quad -d_2 e^{-d_1\tau} I(t), \\ \dot{I}(t) = \beta f'\left(S^*\right) I^* S(t). \end{cases} \tag{5}$$

The parameter satisfy follows conditions:

$$(H1): kr_1 e^{-d_1\tau} > d_1 r_2; 0 < \frac{d_2}{\beta} e^{d_1\tau} < f\left(\frac{kr_1 - d_1 r_2 e^{d_1\tau}}{ke}\right)$$

then the system(1.1)exists unique positive equilibrium point $E^*\left(X^*, S^*, I^*\right)$, where

$$X^* = \frac{r_1 - eS^*}{r_2}, S^* = f^{-1}\left(\frac{d_2}{\beta} e^{d_1\tau}\right), I^* = \frac{\left(kr_1 e^{-d_1\tau} - d_1 r_2 - keS^* e^{-d_1\tau}\right) S^*}{\beta r_2 f(S^*)}$$

the corresponding linear system at equilibrium point $E^*\left(x^*, S^*, I^*\right)$ of (1.1) with form :

$$\dot{u}(t) = \begin{pmatrix} -r_2 X^* & -eX^* & 0 \\ 0 & K - ke^{-d_1\tau} X^* - \beta I^* \left(f'\left(S^*\right) - \dfrac{f(S^*)}{S^*}\right) & -d_2 e^{-d_1\tau} \\ 0 & \beta f'\left(S^*\right) I^* & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 & 0 \\ kS^* e^{-d_1\tau} & \left(kX^* e^{-d_1\tau} - K\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} u(t-\tau) \tag{6}$$

Here, $u(t) = \left(X(t), S(t), I(t)\right)^T$, and the character equation

$$\begin{vmatrix} \lambda + r_2 X^* & eX^* & 0 \\ -ke^{-d_1\tau} S^* e^{\lambda\tau} & \lambda - \left(K - ke^{-d_1\tau} X^* - \beta I^* \left(f'\left(S^*\right) - \dfrac{f(S^*)}{S^*}\right) + \left(kX^* e^{-d_1\tau} - K\right) e^{\lambda\tau}\right) & d_2 e^{-d_1\tau} \\ 0 & -\beta f'\left(S^*\right) I^* & \lambda \end{vmatrix} = 0$$

$$\lambda^3 + \left(r_2 X^* - \left(K - ke^{-d_1\tau} X^* - \beta I^* \left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right)\right)\right)\lambda^2$$

$$+\left(d_2 e^{-d_1\tau}\beta f'\left(S^*\right)I^* - r_2X^*\left(K - ke^{-d_1\tau}X^* - \beta I^*\left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right)\right)\right)\lambda$$

$$+r_2X^*d_2e^{-d_1\tau}\beta f'\left(S^*\right)I^* + \left(-\left(kX^*e^{-d_1\tau} - K\right)\lambda^2 + \left(ke^{-d_1\tau}S^*eX^* - r_2X^*\left(kX^*e^{-d_1\tau} - K\right)\right)\lambda\right)e^{\lambda\tau} = 0$$

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + \left(n_2\lambda^2 + n_1\lambda\right)e^{-\lambda\tau} = 0 \quad \textbf{(7)}$$

where

$$m_2 = \left(r_2X^* - \left(K - ke^{-d_1\tau}X^* - \beta I^*\left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right)\right)\right),$$

$$m_1 = \left(d_2e^{-d_1\tau}\beta f'\left(S^*\right)I^* - r_2X^*\left(K - ke^{-d_1\tau}X^* - \beta I^*\left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right)\right)\right),$$

$$m_0 = r_2X^*d_2e^{-d_1\tau}\beta f'\left(S^*\right)I^*, n_2 = -\left(kX^*e^{-d_1\tau} - K\right), n_1 = \left(ke^{-d_1\tau}S^*eX^* - r_2X^*\left(kX^*e^{-d_1\tau} - K\right)\right);$$

Case1: Special case when $\tau = 0$, the (7) implies that

$$\lambda^3 + \left(m_2 + n_2\right)\lambda^2 + \left(m_1 + n_1\right)\lambda + m_0 = 0 \quad \textbf{(8)}$$

We will give out lemma for positive equilibrium pointto show locally asymptotic stable.

Lemma 3.1 If $(H2)$: $r_2 > ke^{-d_1\tau}, S^* > 1/e; r_2X^* > d_2e^{-d_1\tau}/r_2, f'\left(S^*\right) > f(S^*)/S^*$ holds, then all roots of (8) with the negative real part, and the positive equilibrium point E^* of (2) is locally asymptotic stable.

Proof: First, by $(H2)$,we have $\left(m_2 + n_2\right) = r_2X^* + \beta I^*\left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right) > 0;$

$$\left(m_1 + n_1\right) = ke^{-d_1\tau}S^*eX^* + d_2e^{-d_1\tau}\beta f'\left(S^*\right)I^* + r_2X^*\beta I^*\left(f'\left(S^*\right) - \frac{f(S^*)}{S^*}\right) > 0;$$

$$\left(m_2 + n_2\right)\left(m_1 + n_1\right) - m_0 > 0,$$

under the conditions $\left(H1\right) - \left(H2\right)$, we know by the criterions of $Routh - Hurwits$ the all roots with negative real part of equation (8) and positive equilibrium point E^* of (1) is locally asymptotic stable.

Case 2: when $\tau \neq 0$, we study the general case for (7). We will give following theorem.

Theorem 3.2 If $\left(H1\right), \left(H2\right)$, hold, and suiting one of following conditions:

(1) $\Delta > 0, z^* = \left(\sqrt{\Delta} - p\right)/3 < 0;$

(2) $\Delta > 0, z^* = \left(\sqrt{\Delta} - p\right)/3 > 0, h(z^*) > 0,$

Then this system with that positive equilibrium point E^* of (1)is absolute stability, if $\tau \geq 0$, E^* is also asymptotic stability.

Proof: Assume there exists $\tau^* > 0$, such that the roots of character equation(8)with negative real part. Since $\lambda(\tau)$ is continuous function for τ , there is some $\tau \in \left(0, \tau^*\right)$ such that $\lambda = i\omega(\omega > 0)$ for root of (7). Now, substituting $\lambda = i\omega$ in to (7), and divide the real part and the imaginary part of it,

$$\begin{cases} -n_2\omega^2\cos\omega\tau + n_1\omega\sin\omega\tau = m_2\omega^2 - m_0 \\ n_2\omega^2\sin\omega\tau + n_1\omega\cos\omega\tau = \omega^3 - m_1\omega \end{cases} \quad \textbf{(9)}$$

By square the both side of(18)and additive we obtain:

$$\omega^6 - 2m_1\omega^4 + m_1^2\omega^2 + m_2^2\omega^4 - 2m_2m_0\omega^2 + m_0{}^2 = n_2^2\omega^4 + n_1^2\omega^2$$

Further, we have

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0 \quad (10)$$

where $p = m_2^2 - 2m_1 - n_2^2, q = m_1{}^2 - n_1^2 - 2m_2m_0, r = m_0{}^2.$

Set $z = \omega^2$, then rewrite (10) for that :

$$z^3 + pz^2 + qz + r = 0 \quad (11)$$

and writing $h(z) = z^3 + pz^2 + qz + r,$ defined $\Delta = p^2 - 3q$, combine condition of theorem ,we have that, if $\Delta > 0, z^* = \left(\sqrt{\Delta} - p\right)/3 < 0$; or

and $h(z^*) > 0,$ it is easy to know(10)has no positive root. Thus, we get the conclusion of theorem.

Assume the $(H3): \Delta > 0, z^* = \left(\sqrt{\Delta} - p\right)/3 > 0, h(z^*) < 0,$ hold, by lemma3.2 of literature[11] to know the(10)at least a positive rootthat equation(8)must exist a pair of pure imaginary root : $\pm i\omega_k$. Here, from (9) we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left(\frac{n_2\omega_k^2(m_0 - m_2\omega_k^2) + n_1\omega(\omega_k^3 - m_0\omega_k)}{n_2^2\omega_k^4 + n_1^2\omega_k^2}\right) + 2j\pi\right\}, k = 1, 2, j = 0, 1, 2, \cdots. \quad (12)$$

Lemma 3.3 The real part of eigen-roots satisfies condition:

$$sign\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right)_{\tau=\tau^{(j)}} = sign\left\{\frac{dh\left(\omega_k^2\right)}{dz}\right\}.$$

Proof: Form the both side of (7) by derivative for τ ,we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \sum_{i=1}^{3}\left(\frac{d\lambda}{d\tau}\right)_i^{-1} \quad (13)$$

where

$$\left(\frac{d\lambda}{d\tau}\right)_1^{-1} = \frac{\left(3\lambda^2 + 2m_2\lambda + m_1\right)e^{\lambda\tau}}{\left(n_2\lambda^2 + n_1\lambda\right)\lambda}, \left(\frac{d\lambda}{d\tau}\right)_2^{-1} = \frac{2n_2\lambda + n_1}{\left(n_2\lambda^2 + n_1\lambda\right)\lambda}, \left(\frac{d\lambda}{d\tau}\right)_3^{-1} = -\frac{\tau}{\lambda}$$

By computing, it, and applying (9), check the real part of it, we easy get

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_1^{-1} &= \frac{1}{\Omega}\left\{-\omega_k\left(m_1 - 3\omega_k^2\right)\left(n_1\omega_k\cos\omega_k\tau + n_2\omega_k^2\sin\omega_k\tau\right)\right. \\ &\quad \left. + 2m_2\omega_k^2\left(-n_2\omega_k^2\cos\omega_k\tau + n_1\omega_k\sin\omega_k\tau\right)\right\} \\ &= \frac{1}{\Omega}\left\{3\omega_k^6 + 2\left(m_2^2 - 2m_1\right)\omega_k^4 + \left(m_1^2 - 2m_0m_2\right)\omega_k^2\right\}\end{aligned} \quad (14)$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_2^{-1} = \frac{1}{\Omega}\left\{-n_1^2\omega_k^2 - 2n_2^2\omega_k^4\right\}, \mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_3^{-1} = 0,$$

$$\Omega = n_1{}^2\omega_k^4 + n_2{}^2\omega_k^6 \quad (15)$$

From(14)and(15), we have

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}=\sum_{i=1}^{3}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)_i^{-1}=\frac{1}{\Omega}\left\{3\omega_k^6+2\left(m_2^2-2m_1-n_2^2\right)\omega_k^4+\left(m_1^2-2m_0m_2-n_1^2\right)\omega_k^2\right\}$$

By using of (10)we have

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{1}{\Omega}\left\{3\omega_k^4+2p\omega_k^2+q\right\}\omega_k^2$$

Taking sign of above equality, we get

$$sign\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau^{(j)}}=sign\left\{\frac{dh\left(\omega_k^2\right)}{dz}\right\},\ j=0,1,2,\cdots.$$

By assumption $(H2)$ $dh\left(\omega_k^2\right)/dz\neq 0$, thus transversal condition holdsÿand from existence theorem of Hopf-bifurcation, we know system (1.1) at $\tau=\tau^{(j)},(j=0,1,2,\cdots)$

yields Hopf -bifurcationÿwe have follows conclusion:

Theorem 3.4 If $(H1),(H2)$ hold, we obtain that

(1) As $\tau\in\left[0,\tau^{(0)}\right)$, all root of equation (7) with negative real part, that is locally asymptotically stable at positive equilibrium point E^*;

(2) As $\tau>\tau^{(0)}$, equation(7) has positive real part of root, that isÿunstable at positive equilibrium point E^*;

(3) As $\tau=\tau^{(j)},(j=0,1,2,\cdots)$ are Hopf -bifurcation value of system (1).

THE STABILITY OF EQUILIBRIUM POINT $E_0=(0,0,0),E_1=\left(r_1/r_2,0,0\right)$

Theorem 4.1 We have following conclusion:

(1) For any $\tau_1,\tau\geq 0$, E_0 is a unstable point;

(2) If $\tau=0$ holds: as $K<k\left(r_1/r_2\right)$, then this is locally asymptotical stability at E_1;

As $K>\left(k+kr_1+2r_2\right)r_1/\left(1+r_1\right)r_2$, then this is unstable at E_1.

Proof. It is easy know that characteristic equation of (1):

$$\lambda^3+m_2\lambda^2+m_1\lambda+m_0+\left(n_2\lambda^2+n_1\lambda\right)e^{-\lambda\tau}=0$$

Substituting the value of E_0 into above equation, we easy get $m_2=-K,m_1=0$

$m_0=0,n_2=K,n_1=K$. Then the Character equation of (1.1) at E_0 with that form

$$\lambda^2\left(\lambda-K+Ke^{-\lambda\tau}\right)=0 \quad \textbf{(16)}$$

The equation(15) has two real roots with $\lambda=0$. Notice that if $\lambda\to 0$, then

$-K+Ke^{-\lambda\tau}\to 0$, If $\lambda\to+\infty,-K+Ke^{-\lambda\tau}\to -K$. Therefore, the equation $\lambda-K+Ke^{-\lambda\tau}=0$ has at least one positive real root. Thus, this E_0 is a unstable point.

(2)In the same way, substituting value of E_1 into follows equation

$\lambda^3+m_2\lambda^2+m_1\lambda+m_0+\left(n_2\lambda^2+n_1\lambda\right)e^{-\lambda\tau}=0$, we have

$$m_2=\left(r_1+k\frac{r_1}{r_2}-K\right),m_1=r_1\left(k\frac{r_1}{r_2}-K\right),m_0=0,n_2=-\left(k\frac{r_1}{r_2}-K\right),n_1=-r_1\left(k\frac{r_1}{r_2}-K\right)$$

$$\lambda\left(\lambda^2+\left(r_1+k\frac{r_1}{r_2}-K\right)\lambda+r_1\left(k\frac{r_1}{r_2}-K\right)-\left(k\frac{r_1}{r_2}-K\right)(\lambda+r_1)e^{-\lambda\tau}\right)=0 \quad \textbf{(17)}$$

Next, consider that

$$\lambda^2+p(\tau)\lambda+q(\tau)+\left(r(\tau)+s(\tau)\lambda\right)e^{-\lambda\tau}=0 \quad \textbf{(18)}$$

Here,

$$p(\tau)=\left(r_1+k\frac{r_1}{r_2}-K\right),q(\tau)=r_1\left(k\frac{r_1}{r_2}-K\right),$$

$$r(\tau)=-r_1\left(k\frac{r_1}{r_2}-K\right),s(\tau)=-\left(k\frac{r_1}{r_2}-K\right),$$

When $\tau=0$, from

$$\lambda^2+\left(p(0)+s(0)\right)\lambda+q(0)+r(0)=0 \quad \textbf{(19)}$$

By solving it, we get

$$\lambda_{1,2}=\frac{-\left(p(0)+s(0)\right)\pm\sqrt{\left(p(0)+s(0)\right)^2-4\left(q(0)+r(0)\right)}}{2}$$

$$=\frac{-\left(r_1+(1+r_1)\left(k\frac{r_1}{r_2}-K\right)\right)\pm r_1}{2}=\begin{cases}\dfrac{-(1+r_1)\left(k\frac{r_1}{r_2}-K\right)}{2}\\[2ex]\dfrac{-\left(2r_1+(1+r_1)\left(k\frac{r_1}{r_2}-K\right)\right)}{2}\end{cases}$$

When $K<k\left(r_1/r_2\right)$, we can get that two roots of (10) are negative numbers, $\lambda_{1,2}<0$,this is locally asymptotical stability at equilibrium point E_1; When $K>\left(k+kr_1+2r_2\right)r_1/\left(1+r_1\right)r_2$, we can get that two roots of (10) are positive numbers, $\lambda_{1,2}>0$ ÿit is unstable at equilibrium point E_1; When $K>r_1\left(k(1+r_1)-2r_2\right)/r_2(1+r_1),k>2r_2/(1+r_1)$, we can get that two roots of (10) are a positive numbers and a negative number, $\lambda_1>0,\lambda_2<0$, we can get that it is a unstable point at equilibrium E_1 .we complete the proof.

DIRECTION OF BIFURCATION AND COMPUTING FORMULA FOR BIFURCATION OF PERIODIC SOLUTION

By normal form method and center manifold theorem, we give that direction of Hopf-bifurcation of (1), and stability of bifurcation for periodic solution and periodic computing formula of (1)

$$\dot{u}(t)=\begin{pmatrix} -r_2X^* & -eX^* & 0 \\ 0 & K-ke^{-d_1\tau}X^*-\beta I^*\left(f'(S^*)-\frac{f(S^*)}{S^*}\right) & -d_2e^{-d_1\tau} \\ 0 & \beta f'(S^*)I^* & 0 \end{pmatrix}u(t)$$
$$+\begin{pmatrix} 0 & 0 & 0 \\ kS^*e^{-d_1\tau} & \left(kX^*e^{-d_1\tau}-K\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}u(t-\tau)$$

Where $u(t)=\left(X(t),S(t),I(t)\right)^T$.

Let: $u_1=X-X^*, u_2=S-S^*, u_3=I-I^*, \overline{u}_i(t)=u_i(\tau t), \tau=\tau_0+v;$ then system (1) may write

$$\dot{u}(t)=L_\mu(u_t)+f(\mu,u_t), \tag{20}$$

where: $u(t)=\left(u_1(t),u_2(t),u_3(t)\right)^T\in R^3, u_t(\theta)=u(t+\theta), L_v:C\to R^3, f:R\times C\to R^3$ may can corresponding expresses follow :

$$L_\mu(u_t)=(\tau_0+\mu)\begin{pmatrix} -r_2X^* & -eX^* & 0 \\ 0 & K-ke^{-d_1\tau_1}X^*-\beta I^*\left(f'(S^*)-\frac{f(S^*)}{S^*}\right) & -d_2e^{-d_1\tau_1} \\ 0 & \beta f'(S^*)I^* & 0 \end{pmatrix}\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\end{pmatrix}$$
$$+(\tau_0+\mu)\begin{pmatrix} 0 & 0 & 0 \\ ke^{-d_1\tau_1}S^* & \left(kX^*e^{-d_1\tau_1}-K\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\end{pmatrix}, \tag{21}$$

and

$$f(\mu,u_t)=(\tau_0+\mu)\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}=(\tau_0+\mu)\begin{pmatrix} -r_2\phi_1^{\,2}(0)-e\phi_1(0)\phi_2(0) \\ k\phi_1(-1)\phi_2(-1)-\Gamma 1 \\ \beta e^{-\gamma\tau}f'(S^*)\phi_2(0)\phi_3(0)+\Gamma 2 \end{pmatrix}, \tag{22}$$

Here,

$\Gamma 1=$

$$\frac{\beta e^{-d_1\tau}}{6}\left(3f''(S^*)\phi_2^2(0)+2f'''(S^*)\phi_2^3(0)+3f''(S^*)\phi_2^2(0)\phi_3(0)+2f'''(S^*)\phi_2^3(0)\phi_3(0)\right)+O(4),$$

$\Gamma 2 =$

$$\frac{\beta e^{-d_1\tau}}{6}\left(3f''\left(S^*\right)I^*\phi_2^2(0)+2f'''\left(S^*\right)I^*\phi_2^3(0)+3f''\left(S^*\right)\phi_2^2(0)\phi_3(0)+2f'''\left(S^*\right)\phi_2^3(0)\phi_3(0)\right)+O(4)$$

By therepresentation theorem, there existence a function $\eta(\theta,\mu)$, with bounded variation for $\theta\in[-1,0]$, such that

$$L_\mu(\phi)=\int_{-1}^{0} d\eta(\theta,0)\phi(\theta),\ \text{for}\ \phi\in C.$$

Indeed, it be chosen

$$\eta(\theta,\mu)=(\tau_0+\mu)A\delta(\theta)-(\tau_0+\mu)B\delta(\theta+1), \tag{23}$$

where δ expresses $Dirac-delta$ function $(Dirac-delta)$. For $\phi\in C\left([-1,0],R^3\right)$, define

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{d\phi(\theta)}{d\theta},\theta\in[-1,0),\\ \int_{-1}^{0} d\eta(s,\mu)\phi(s),\theta=0.\end{cases}$$

And

$$R(\mu)\phi=\begin{cases}0,\theta\in[-1,0),\\ f(\mu,\phi),\theta=0.\end{cases}$$

As $\theta=0$, (20) equivalent

$$\dot{u}(t)=A(\mu)u_t+R(\mu)u_t, \tag{24}$$

Here, $u(t)=\left(u_1(t),u_2(t)\right)^T$, $u_t(\theta)=u(t+\theta),\theta\in[-1,0)$.

For $\psi\in C^1\left([-1,0],\left(R^3\right)^*\right)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds},s\in(-1,0],\\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t),s=0.\end{cases}$$

And bilinear product:

$$\langle\psi(s),\phi(\theta)\rangle=\bar{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{25}$$

where, $\eta(\theta)=\eta(\theta,0)$.°‹ $A=A(0)$, then A and A^* are adjoint operator.

Lemma 5.1 Vector

$$q(\theta)=(1,q_1,q_2)^T e^{i\omega_0\tau_0\theta}, q^*(s)=D\left(1,q_1^*,q_2^*\right)e^{i\omega_0\tau_0 s}$$

are two character vector for the $i\omega_0\tau_0$ is eigen value of $A(0)$ and the $-i\omega_0\tau_0$ of A^*,

respectively, and $\langle q^*(s), q(\theta)\rangle = 1, \langle q^*(s), \bar{q}(\theta)\rangle = 0,$ where

$$\bar{D} = \frac{1}{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - \left(ke^{-d_1\tau_1}S^* + \left(kX^*e^{-d_1\tau_1} - K\right)q_1\right)\bar{q}_1^* e^{-i\omega_0\tau_0}}.$$

Proof. Assume $q(\theta), q^*(s)$ are Character vector $A(0)$ for $i\omega_0$ and A^* for $-i\omega_0$ (from above section discussion), respectively, then we have $Aq(\theta) = i\omega_0\tau_0 q(\theta)$. From definition of A and (24), we obtain that satisfy

$$\tau_0\begin{pmatrix} i\omega_0 + r_2X^* & eX^* & 0 \\ 0 & i\omega_0 + ke^{-d_1\tau}X^* + \beta I^*\left(f'(S^*) - \dfrac{f(S^*)}{S^*}\right) - K & d_2e^{-d_1\tau_1} \\ 0 & -\beta f'(S^*)I^* & i\omega_0 \end{pmatrix}\begin{pmatrix}1\\ q_1\\ q_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

By calculating, we get

$$q(0) = (1, q_1, q_2)^T = \left(1, -\frac{(r_2X^* + i\omega_0)}{eX^*}, -\frac{\beta e^{-d_1\tau_1}f'(S^*)I^*q_1}{i\omega_0}\right)^T$$

In the same way,

$$q_1^* = \frac{eX^*}{k - \left(ke^{-d_1\tau_1}X^* + \beta\left(f'(S^*) - f(S^*)/S^*\right)I^* - K\right) + i\omega_0}, q_2^* = \frac{d_2I^*q_1^*}{i\omega_0};$$

Next, finding that.

$$\begin{aligned}\langle q^*(S), q(\theta)\rangle &= \bar{D}\left(1, \bar{q}_1^*, \bar{q}_2^*\right)(1, q_1, q_2)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}\left(1, \bar{q}_1^*, \bar{q}_2^*\right)e^{-i(\xi-\theta)\omega_0\tau_0}d\eta(\theta)(1, q_1, q_2)^T e^{i\xi\omega_0\tau_0}d\xi \\ &= \bar{D}\left\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - \int_{-1}^{0}\left(1, \bar{q}_1^*, \bar{q}_2^*\right)\theta e^{i\theta\omega_0\tau_0}d\eta(\theta)(1, q_1, q_2)^T\right\} \\ &= \bar{D}\left\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - \left(ke^{-d_1\tau_1}S^* + \left(kX^*e^{-d_1\tau_1} - K\right)q_1\right)\bar{q}_1^*e^{-i\omega_0\tau_0}\right\}.\end{aligned}$$

Therefore, choose

$\bar{D} = \left\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - \left(kS^* + \left(kX^* - K\right)q_1\right)q_1^*e^{-i\xi\omega_0\tau_0}\right\}^{-1}$, such that $\langle q^*(s), q(\theta)\rangle = 1$.

In the same method, we obtain

$$\begin{aligned}\langle q^*(S), \bar{q}(\theta)\rangle &= \bar{D}\left(1, \bar{q}_1^*, \bar{q}_2^*\right)(1, \bar{q}_1, \bar{q}_2)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{D}\left(1, \bar{q}_1^*, \bar{q}_2^*\right)e^{-i(\xi-\theta)\omega_0\tau_0}d\eta(\theta)(1, \bar{q}_1, \bar{q}_2)^T e^{i\xi\omega_0\tau_0}d\xi \\ &= 0.\end{aligned}$$

Following the algorithms explained in[23] and using a computation process similar to that in[20], which is used to obtain the properties of Hopf bifurcation:

$$g_{20} = 2\bar{D}\tau_0\left\{-r_2 - eq_1 + k\bar{q}_1^* q_2 e^{-2i\omega_0\tau_0} + \beta q_1 q_2 \bar{q}_2^* - \frac{1}{2}\beta\bar{q}_1^* q_1^2 f''(S^*)e^{-d_1\tau_1} + \frac{1}{2}\beta\bar{q}_2^* q_1^2 f''(S^*)I^* e^{-d_1\tau_1}\right\},$$

$$g_{11} = 2\bar{D}\tau_0\left\{-2r_2 - eq_1 - e\bar{q}_1 + k\bar{q}_1^*\bar{q}_2 + k\bar{q}_1^* q_2 + \beta\bar{q}_2^*\bar{q}_1 q_2 + \beta\bar{q}_2^*\bar{q}_2 q_1 + \beta\bar{q}_2^*\bar{q}_1 q_1 f''(S^*)e^{-d_1\tau_1}\right\},$$

$$g_{02} = 2\bar{D}\tau_0\left\{-0.5e - r_2 - e\bar{q}_1 + k\bar{q}_1^*\bar{q}_2 e^{2i\omega_0\tau_0} + \beta\bar{q}_2^*\bar{q}_1\bar{q}_2 - \beta\bar{q}_1^* q_1\bar{q}_1 f''(S^*)e^{-d_1\tau_1} - \frac{1}{2}\beta\bar{q}_1^*\bar{q}_1^2 f''(S^*)e^{-d_1\tau_1}\right.$$
$$\left. + \frac{1}{2}\beta\bar{q}_2^*\bar{q}_1^2 f''(S^*)I^* e^{-d_1\tau_1}\right\},$$

$$g_{21} = 2\bar{D}\tau_0\left\{-\left(r_2 + \frac{1}{2}e\right)W_{20}^{(1)}(0) - (2r_2 + e)W_{11}^{(1)}(0)\right.$$
$$-\left(e + \beta\bar{q}_1^* q_1 f''(S^*)e^{-d_1\tau_1} - \beta\bar{q}_2^* q_1 f''(S^*)I^* e^{-d_1\tau_1}\right)W_{11}^{(2)}(0)$$
$$-\frac{1}{2}\left(e + \beta\bar{q}_1^*\bar{q}_1 f''(S^*)e^{-d_1\tau_1} - \frac{1}{2}\beta\bar{q}_2^*\bar{q}_2 - \frac{1}{2}\beta\bar{q}_2^*\bar{q}_1 f''(S^*)I^* e^{-d_1\tau_1}\right)W_{20}^{(2)}(0)$$

$$+k\bar{q}_1^* e^{-i\omega_0\tau_0}W_{11}^{(2)}(-1) + \frac{1}{2}k\bar{q}_1^* e^{i\omega_0\tau_0}W_{20}^{(2)}(-1) - \frac{1}{2}k\bar{q}_1^*\bar{q}_2 e^{i\omega_0\tau_0}W_{20}^{(1)}(-1)$$
$$+k\bar{q}_1^* q_2 e^{-i\omega_0\tau_0}W_{11}^{(1)}(-1) - \beta\bar{q}_1^*\bar{q}_1 q_1^2 f'''(S^*)e^{-d_1\tau_1} - \frac{1}{2}\beta\bar{q}_1^* q_1^2\bar{q}_2 f''(S^*)e^{-d_1\tau_1}$$
$$-\beta\bar{q}_2^* q_1\bar{q}_1 q_2 f''(S^*)e^{-d_1\tau_1} + \beta\bar{q}_2^* q_1^2\bar{q}_1 I^* f'''(S^*)e^{-d_1\tau_1} + \frac{1}{2}\beta\bar{q}_2^* q_1^2\bar{q}_2 f''(S^*)e^{-d_1\tau_1} \quad \textbf{(26)}$$
$$\left.+\beta\bar{q}_2^* q_1\bar{q}_1 f'''(S^*)e^{-d_1\tau_1} + \frac{1}{2}\beta\bar{q}_2^*\bar{q}_1 W_{20}^{(3)}(0) + \beta\bar{q}_2^* q_2 W_{11}^{(2)}(0) + \beta\bar{q}_2^* q_1 W_{11}^{(3)}(0)\right\}$$

Where

$$\begin{cases} W_{20}(\theta) = \dfrac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \dfrac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}, \\ W_{11}(\theta) = -\dfrac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \dfrac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2 \end{cases} \quad \textbf{(27)}$$

here $E_i = \left(E_i^{(1)}, E_i^{(2)}, E_i^{(3)}\right) \in R^3, i = 1, 2,$ for constant vector.

Further, by calculating with the first equation of(27), we get

$$\begin{pmatrix} E_{11} & eX^* & 0 \\ 0 & E_{22} & d_2 e^{-d_1\tau_1} \\ 0 & -\beta f'(S^*)I^* & E_{33} \end{pmatrix} E_1 = 2\begin{pmatrix} -r_2 - eq_1 \\ kq_1^* q_2 e^{-2i\omega_0\tau_0} - \dfrac{1}{2}\beta\bar{q}_1^* q_1^2 f''(S^*)e^{-d_1\tau_1} \\ \beta q_1 q_2\bar{q}_2^* \dfrac{1}{2}\beta\bar{q}_2^* q_1^2 f''(S^*)I^* e^{-d_1\tau_1} \end{pmatrix}.$$

Here

$$E_{11} = 2i\omega_0 + r_2 X^*, E_{22} = 2i\omega_0 + \left(ke^{-d_1\tau} X^* + \beta I^* \left(f'\left(S^*\right) - \frac{f\left(S^*\right)}{S^*} \right) - K \right), E_{33} = 2i\omega_0$$

$$E_1^{(1)} = \frac{2}{A} \begin{vmatrix} -r_2 - eq_1 & eX^* & 0 \\ kq_1^* q_2 e^{-2i\omega_0\tau_0} - \frac{1}{2}\beta \bar{q}_1^* q_1^2 f''(S^*) e^{-d_1\tau_1} & E_{22} & d_2 e^{-d_1\tau_1} \\ \beta q_1 q_2 \bar{q}_2^* + \frac{1}{2}\beta \bar{q}_2^* q_1^2 f''(S^*) I^* e^{-d_1\tau_1} & -\beta f'\left(S^*\right) I^* & E_{33} \end{vmatrix},$$

$$E_1^{(2)} = \frac{2}{A} \begin{vmatrix} E_{11} & -r_2 - eq_1 & 0 \\ 0 & kq_1^* q_2 e^{-2i\omega_0\tau_0} - \frac{1}{2}\beta \bar{q}_1^* q_1^2 f''(S^*) e^{-d_1\tau_1} & -d_2 e^{-d_1\tau_1} \\ 0 & \beta q_1 q_2 \bar{q}_2^* + \frac{1}{2}\beta \bar{q}_2^* q_1^2 f''(S^*) I^* e^{-d_1\tau_1} & E_{33} \end{vmatrix},$$

$$E_1^{(3)} = \frac{2}{A} \begin{vmatrix} E_{11} & eX^* & -r_2 - eq_1 \\ 0 & E_{22} & kq_1^* q_2 e^{-2i\omega_0\tau_0} - \frac{1}{2}\beta \bar{q}_1^* q_1^2 f''(S^*) e^{-d_1\tau_1} \\ 0 & -\beta f'\left(S^*\right) I^* & \beta q_1 q_2 \bar{q}_2^* + \frac{1}{2}\beta \bar{q}_2^* q_1^2 f''(S^*) I^* e^{-d_1\tau_1} \end{vmatrix}$$

where

$$A = \begin{vmatrix} E_{11}^{\ *} & eX^* & 0 \\ 0 & E_{22} & -d_2 e^{-d_1\tau_1} I^* \\ 0 & -\beta f'\left(S^*\right) I^* & E_{33} \end{vmatrix}.$$

In the same way, from the second equation of (27) that we have

$$\begin{pmatrix} r_2 X^* & eX^* & 0 \\ 0 & E_{22} - 2\omega_0 i & d_2 e^{-d_1\tau_1} \\ 0 & -\beta f'\left(S^*\right) I^* & 0 \end{pmatrix} E_2 = 2 \begin{pmatrix} -2r_2 - 2e\,\mathrm{Re}\left\{q_1\right\} \\ 2k\,\mathrm{Re}\left\{q_2\right\} \\ \beta\,\mathrm{Re}\left\{\bar{q}_1 q_2\right\} + \beta\,\mathrm{Re}\left\{\bar{q}_2 q_1\right\} + \beta \bar{q}_1 q_1 f''(S^*) e^{-d_1\tau_1} \end{pmatrix}$$

Moreover, by computing it, we obtain

$$E_2^{(1)} = \frac{2}{B} \begin{vmatrix} -2r_2 - 2e\,\mathrm{Re}\left\{q_1\right\} & eX^* & 0 \\ 2k\,\mathrm{Re}\left\{q_2\right\} & E_{22} - 2\omega_0 i & d_2 e^{-d_1\tau_1} \\ \beta\,\mathrm{Re}\left\{\bar{q}_1 q_2\right\} + \beta\,\mathrm{Re}\left\{\bar{q}_2 q_1\right\} + \beta \bar{q}_1 q_1 f''(S^*) e^{-d_1\tau_1} & -\beta f'\left(S^*\right) I^* & 0 \end{vmatrix}$$

$$E_2^{(2)} = \frac{2}{B}\begin{vmatrix} r_2X^* & -2r_2 - 2e\operatorname{Re}\{q_1\} & 0 \\ 0 & 2k\operatorname{Re}\{q_2\} & d_2e^{-d_1\tau_1} \\ 0 & \beta\operatorname{Re}\{\bar{q}_1 q_2\} + \beta\operatorname{Re}\{\bar{q}_2 q_1\} + \beta\bar{q}_1 q_1 f''(S^*)e^{-d_1\tau_1} & 0 \end{vmatrix}$$

$$E_2^{(3)} = \frac{2}{B}\begin{vmatrix} r_2X^* & eX^* & -2r_2 - 2e\operatorname{Re}\{q_1\} \\ 0 & E_{22} - 2\omega_0 i & 2k\operatorname{Re}\{q_2\} \\ 0 & -\beta f'(S^*)I^* & \beta\operatorname{Re}\{\bar{q}_1 q_2\} + \beta\operatorname{Re}\{\bar{q}_2 q_1\} + \beta\bar{q}_1 q_1 f''(S^*)e^{-d_1\tau_1} \end{vmatrix}$$

here

$$B = \begin{vmatrix} r_2X^* & eX^* & 0 \\ 0 & E_{22} - 2\omega_0 i & d_2e^{-d_1\tau_1} \\ 0 & -\beta f'(S^*)I^* & 0 \end{vmatrix}.$$

Substituting E_1, E_2 into (27), then we get g_{21}. So we can compute these qualityÿ

$$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}},$$
$$T_2 = \frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}, \qquad \beta_2 = 2\operatorname{Re}\{C_1(0)\}. \quad \textbf{(28)}$$

Hence, we have following theorem,
Theorem 3 Assume that

(1) μ_2 decide the direction of Hopf-bifurcationÿIf $\mu_2 > 0(<0)$, *Hopf* -bifurcation for with upper critical (lower critical);

(2) μ_2 decide the stable properties of Hopf-bifurcation, If $\beta_2 < 0(>0)$, this bifurcation with periodic solution is with orbit asymptotically stable (unstable);

(3) T_2 diced the periodic of periodic solution for Hopf-bifurcation. If $T_2 > 0(<0)$, this periodic is monotone increase along to τ (monotone decreasing).

NUMERICAL SIMULATION

In this section, we check nonlinear function of incidence $f(s) = s^2$, to give several class parameters by using of previous computing method and Matlab to get stability of fixed point for system (1). The abased by use of linear term of feedback control $K(S(t) - S(t-\tau))$, and this term holds action for two delay $\beta e^{d_1\tau_1} S(t-\tau)X(t-\tau)$ and occurs rate, combine parameters of (1) to discuss dynamic action of (1).

(1) Suppose factor of feedback control $K = 0$, choose as follows parameters

r1	r2	beta	e	k	d1	d2	tau
1	1.2	1	1.5	1.2	0.18	0.25	0.2

We may compute for $E^*\left(0.1970\ 0.5091\ 0.1148\right)$, $\omega^* = 0.2953, \tau^* = 3.9879$ ÿand from theorem2 know the positive equilibrium E^* is locally asymptotic stability, Figure 1 by(a)(b), and from theorem2 know at neighborhood of positive equilibrium E^* yields bifurcating periodic solution as Figure 2 by(a)(b) expresses.

(2) When $K \neq 0, \tau \neq 0, \tau_1 = 0.2$, choose as parameters

r1	r2	beta	e	k	d1	d2
1	1.2	1	1.5	1.2	0.1	0.2

We will discuses at E^* yields conditions for Hopf-bifurcation, if $K \neq 0, \tau \neq 0$, Let $i\omega$ is root of characteristic equation $\lambda^3 + p\lambda^2 + q\lambda + r = 0$, it must suite bellow equation

$$z^3 + \left(0.1667 + 0.6449K - \left(K - 0.3161\right)^2\right) z^2 + \left(\left(0.2369 - 0.3224K\right)^2 - 0.0382 - \left(0.1122 + 0.3224K\right)^2\right) z + 0.0006 = 0 \quad (*)$$

where $z = \omega^2$. Set $p = 0.1667 + 0.6449K - \left(K - 0.3161\right)^2$,

$q = \left(0.2369 - 0.3224K\right)^2 - \left(0.1122 + 0.3224K\right)^2 - 0.0382$ and $r = 0.0006$, Then we have

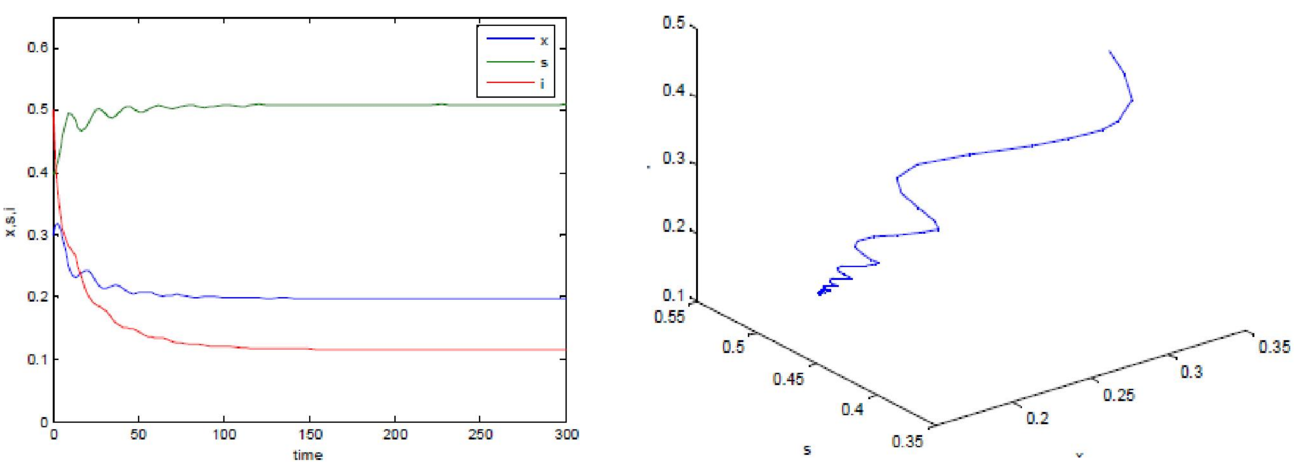
$\Delta = p^2 - 3q = K^2 + 35.5729K - 304.2539$. So, we have $\Delta \geq 0$ if and only if $K < -0.3498$

or $K > 0.0132$. From above Theorem 3.2, we have $z_1^* = (-p + \sqrt{\Delta})/3 > 0, \Delta = p^2 - 3q \geq 0$;

when $K > 0.3812$, the equation (*) has positive root.

(1) Taking $K = 2$, $K \neq 0, \tau \neq 0, \tau_1 = 0.2, \tau_1 \neq \tau$, We may compute for $E^*(0.2687, 0.4517, 0.5024)$,

$\omega^* = 1.4955, \tau_1^* = 0.4562$. By lemma3.3 and from formula of calculating method for (5.9), we have



(a) implies wave form of(1) (b) implies images of(1)

Figure 1 : When $\tau = 2 < \tau^*$, the positive equilibrium E^* is asymptotically stable

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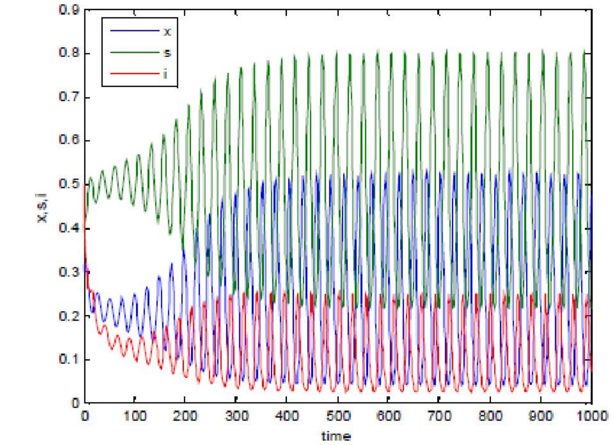
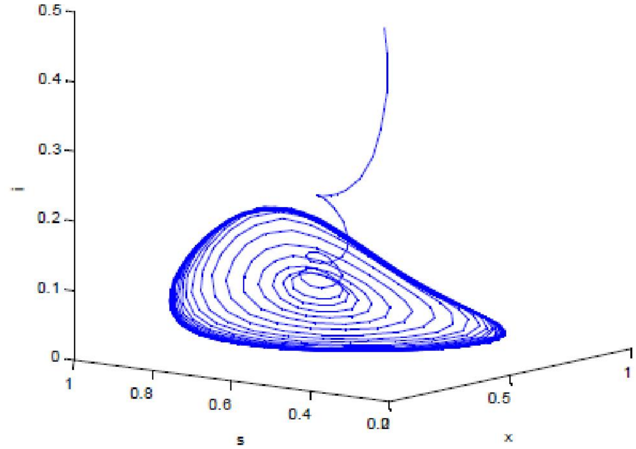


(a) implies wave form of(1)



(b) expresses images of(1)

Figure 2 : When $\tau = 5.5 > \tau^*$, bifurcating periodic solution fromoccur.

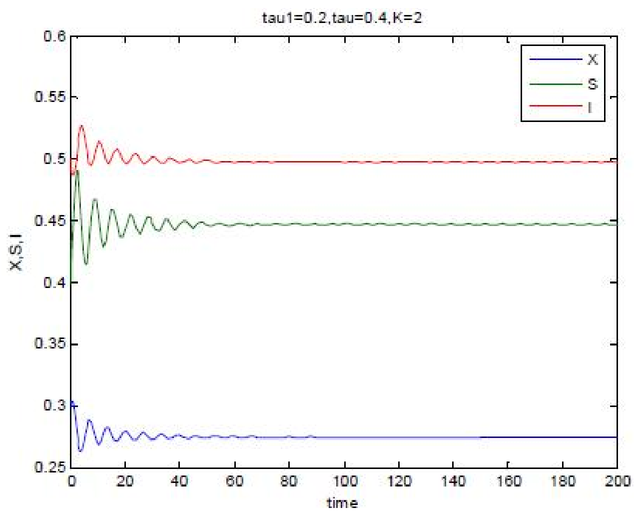


Figure 3 : (a) implies wave form of(6.1).

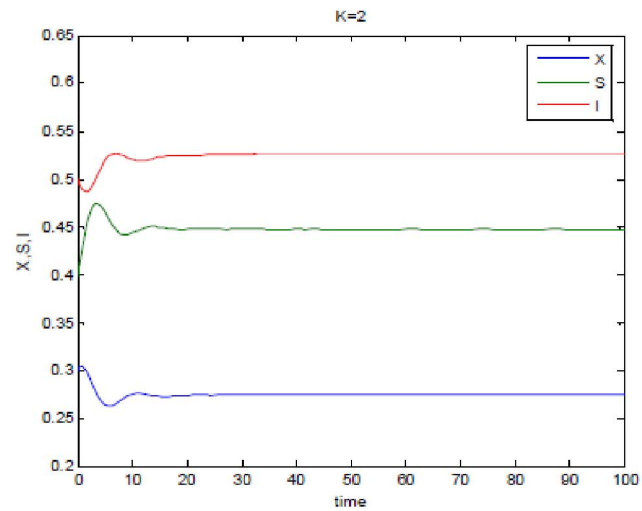
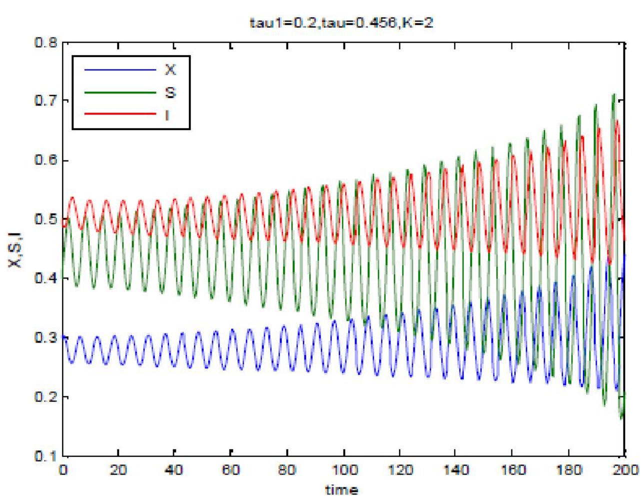
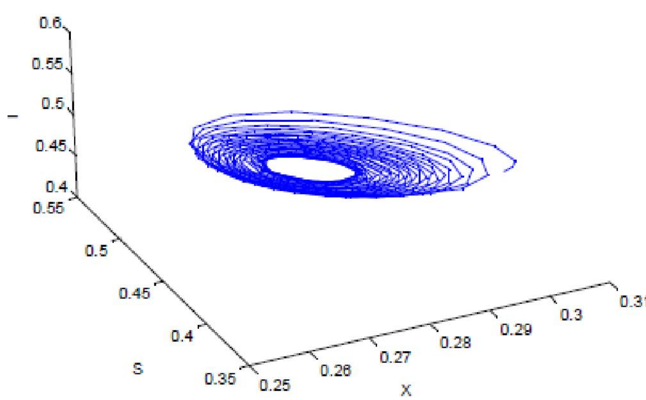


Figure 3 : (b) implies wave of(6.2).



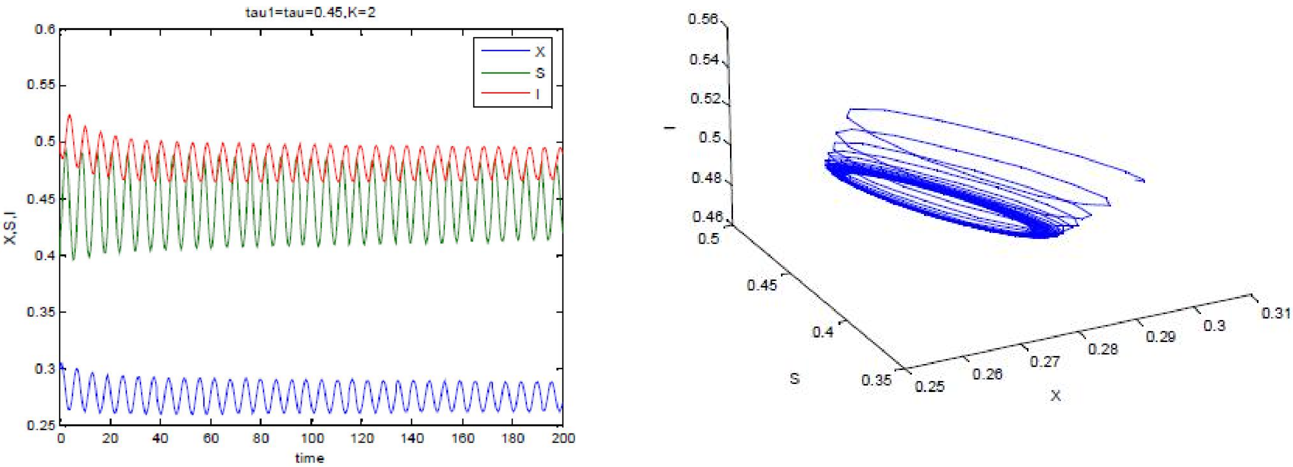
(a) implies wave form of(6.1).



(b) expresses images of(6.1).

Figure 4 : When bifurcating periodic solution fromoccur.

that as $\tau = \tau_1^*$ these value of parameters $\operatorname{Re} C_1(0) < 0$,



(a) implies wave form of(6.2) **(b) expresses images of (6.2)**

Figure 5 : When bifurcating periodic solution fromoccur

$$\operatorname{Re}\left\{\frac{\lambda(\tau_1^*)}{d\tau}\right\} = \frac{n_1^2\omega^2 + n_2^2\omega^4}{3\omega^4 + 2p\omega^2 + q} = 1.8639 > 0, Sign \operatorname{Re}\left\{\frac{\lambda(\tau_1^*)}{d\tau}\right\} > 0, \qquad \mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re}\lambda(\tau_1^*)} > 0$$ and

$\beta_2 = 2\operatorname{Re} C_1(0) < 0$, we may get stability for positive equilibrium point (See Figure 3 (a)) of bellow system

$$\begin{cases} \dot{X} = X(t)(1-1.2X(t)) - 1.5X(t)S(t), \\ \dot{S}(t) = 1.2e^{-0.02\tau_1}S(t-\tau)x(t-\tau) - S^2(t)I(t) - 0.1S(t) + 2(S(t) - S(t-\tau)), \\ \dot{I}(t) = S^2(t)I(t) - 0.2\cdot I(t). \end{cases} \quad \textbf{(28)}$$

and hold conclusion for periodic solution (See Figure 4(a),(b)).

(2) Taking $K = 2$, $K \neq 0, \tau = \tau_1 = 0.45$ ÿWe may compute for $E^*(0.2616\ 0.4574\ 0.4892)$, $\omega^* = 1.5148, \tau_1^* = 0.4500$. By lemma3.3 and from formula of calculating method for (27), we have that as $\tau = \tau_1^*$, these value of parameters $\operatorname{Re} C_1(0) < 0$,

$\operatorname{Re}\left\{\frac{\lambda(\tau_1^*)}{d\tau}\right\} = 1.8639 > 0, Sign \operatorname{Re}\left\{\frac{\lambda(\tau_1^*)}{d\tau}\right\} > 0$ and $\beta_2 = 2\operatorname{Re} C_1(0) < 0$ ÿwe may get stability for positive equilibrium point (See Figure 3.(b))of bellow system

$$\begin{cases} \dot{X} = X(t)(1-1.2X(t)) - 1.5X(t)S(t), \\ \dot{S}(t) = 1.2e^{-0.02\tau}S(t-\tau)x(t-\tau) - S^2(t)I(t) - 0.1S(t) + 2(S(t) - S(t-\tau)), \\ \dot{I}(t) = S^2(t)I(t) - 0.2\cdot I(t). \end{cases} \quad \textbf{(28)}$$

and hold conclusion for periodic solution(See Figure 5. (a),(b)).

CONCLUSIONS

In this paper, an eco-epidemiology model with nonlinear incidence and susceptible (the predator population) pregnancy number delay of is formulated.

Sufficient conditions of local stability of the positive equilibrium arc given by analyzing the characteristic value of linear system (1) and conditions ensuring the existence of Hopf-bifurcation are obtained. It is proved that when the delayis suitable small, the positive equilibrium is locally stability and the Hopf-bifurcations occur when thepasses through a sequence of critical values (such as).

In the literature, however, the system (1), there are few research. This paper from two aspects of theory and numerical discussion for ecological-epidemic model with delay, as you can see, the numerical results and the theoretical research are same meaning, so as to verify the validity of the theoretical for derivation. Through the discussion of this paper, not only can reflect the behavior of the complexity of the system dynamics, but also deepened with the feedback of the general time-delay control ecological-epidemic model corresponds for its dynamic behavior.

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