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Relativistic spinless particles with generalized exponential potential

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Abstract

In this paper, we present the solutions of the Klein-Gordon equation for a generalized exponential potential field by applying the Pekeris approximation to the centrifugal term. We used the parametric generalization of the Nikiforov-Uvarov method to obtain the energy eigenvalues and the corresponding wave function in a closed form. Special cases of the potential are also discussed.

Key Words

Klein-Gordon equation; Nikiforov-Uvarov method; Generalized exponential potential.

INTRODUCTION

In both relativistic and non-relativistic quantum mechanics, the exact solutions play an important role since they contain all the necessary information regarding the quantum system under consideration[1]. The Klein-Gordon (KG) equation is the well-known relativistic wave equation describing spin-zero particles due to its square terms and in many cases possessed solutions already known through solutions of Schrödinger equations. Consequently, the square term in the KG equation causes a lot of complexity for some potential especially when unequal scalar and vector potentials are studied[2,3]. However, the analytical solutions of the KG equations are possible for a few simple cases such as hydrogen atom, the harmonic oscillator and others[4,5]. Therefore with the used of various technique such as Nikiforov-Uvarov (NU) method[6-8], Supersymmetric quantum mechanics (SUSSY)[9,10], the Lie algebraic approach[11], the point canonical transformation (PCT)[12], asymptotic iteration method (AIM)[13], the shifted $\frac{1}{N}$ expansion (SE) technique[14], the exact quantization rule[15], the polynomial expansion[16], the Pekeris approximation[17], perturbation theory[18], the ansatz approach[19], a number of quantum mechanical systems for arbitrary *l*-states have been studied by taking a good or elegant approximation to the centrifugal term within the non-relativistic and relativistic wave equations. Quite interesting, thee relativistic wave equation namely, the KG equations have been solved for various potentials model including the Hulthen[20], Rosen–Morse[21], Poschl-Teller[22], Hylleraas[23,24], exponential-type[25], Deng-Fan[26] and others[27].

In this paper, we consider in d-dimensional case a more generalized exponential-type potential which provided us with a more exact results in comparison with other potentials model.

HYPERRADIAL PART IN D-DIMENSIONS

It is well-known that the radial part of KG equations in the presence of vector and scalar potentials in the D-dimensional space can be written as,

$$\left\{\begin{matrix}\frac{d^2}{dr^2}+E_{n,l}^2+V^2(r)-2E_{n,l}V(r)-m^2(r)-S^2(r)-2m(r)S(r)\\-\frac{(D+2l-1)(D+2l-3)}{4r^2}\end{matrix}\right\}U_{n,l}(r)=0 \quad \textbf{(1)}$$

Here consider a generalized exponential-type potential of the form,

$$V(r)=V_0\left(A+Be^{-\alpha(r-r_0)}\right), S(r)=S_0\left(A+Be^{-\alpha(r-r_0)}\right), \quad (2)$$

Where r denotes the hyperradius and A, B, V_0, S_0 and α are constant coefficients. When the constant A = 0 and B = -1, the potential model reduces to the exponential-type potential reported by Hassanabadi et al[25]. Also, when V_0 = D_e, A = -1, B = 1 and map α → 2a, the generalized exponential-type potential model reduces to the well-known Morse potential[21]. We also consider a position-dependent mass term of the form,

$$m(r)=m_0+m_1\left(A+Be^{-\alpha(r-r_0)}\right), \quad (3)$$

Where C, D′ are constant coefficients. Substituting Eqs.(2) and (3) into Eq.(1), we obtain,

$$\left.\begin{array}{l}\frac{d^2}{dr^2}+E_{n,l}^2+V_o^2\left(A+Be^{-\alpha(r-r_0)}\right)^2-2E_{n,l}V_0\left(A+Be^{-\alpha(r-r_0)}\right)\\-\left(m_0+m_1\left(A+Be^{-\alpha(r-r_0)}\right)\right)^2\\-S_0^2\left(A+Be^{-\alpha(r-r_0)}\right)^2-2S_0\left(m_0+m_1\left(A+Be^{-\alpha(r-r_0)}\right)\right)\\\left(A+Be^{-\alpha(r-r_0)}\right)-\frac{(D+2l-1)(D+2l-3)}{4r^2}\end{array}\right\}U_{n,l}(r)=0 \quad (4)$$

Equation (4) can not be solved analytically for $l\neq 0$, because of the angular momentum term. Therefore, we shall use the Pekeris approximation[17] in order to deal with the centrifugal term. In the Pekeris approximation the centrifugal term is expanded around r = r_0 in a series of power of x = (r - r_0)/r_0 as,

$$\frac{1}{r^2}=\frac{1}{r_0^2(1+x)^2}=\frac{1}{r_0^2}\left(1-2x+3x^2-4x^3+...\right) \quad (5)$$

Thus the Pekeris approximation[17] can be expanded for the centrifugal term as,

$$\frac{1}{r^2}=\frac{1}{r_0^2}\left(C_0+C_1e^{-ax}+C_2e^{-2ax}\right), \quad (6)$$

Where α = ar_0, C_i is the parameter coefficients (i = 0,1,2) and expanding Eq.(6) up to four term, we get

$$\frac{1}{r^2}=\frac{1}{r_0^2}\left(C_0+C_1\left(1-\alpha x+\frac{\alpha^2x^2}{2!}-\frac{\alpha^3x^3}{3!}+...\right)+C_2\left(1-2\alpha x+\frac{4\alpha^2x^2}{2!}-\frac{8\alpha^3x^3}{3!}+...\right)\right) \quad (7)$$

Arranging Eq.(7) and comparing equal powers with Eq.(6), we obtain the relations between the coefficients and the parameter α as,

$$\boxed{C_0=\frac{1}{r_0^2}\left(1-\frac{3}{\alpha}+\frac{3}{\alpha^2}\right)}, C_1=\frac{1}{r_0^2}\left(\frac{4}{\alpha}-\frac{6}{\alpha^2}\right), C_2=\frac{1}{r_0^2}\left(\frac{-1}{\alpha}+\frac{3}{\alpha^2}\right) \quad (8)$$

Substituting Eq.(6) into Eq.(4), we obtain,

$$\left\{\frac{d^2}{dr^2}+r_0^2\left(p_0+p_1e^{-\alpha r}+p_2e^{-2\alpha r}\right)\right\}U_{n,l}(r)=0, \quad (9)$$

Where,

$$\begin{array}{l}p_0=E_{n,l}^2-m_0^2+\left(V_0^2-S_0^2-m_1^2-2S_0S_1\right)A^2\\\quad-2\left(E_{n,l}V_0+S_0m_0+m_0m_1\right)A-D_0,\\D_0=\frac{(D+2l-1)(D+2l-3)C_0}{4}\\p_1=2\left(V_0^2-S_0^2-m_1^2-m_1S_0\right)AB\\\quad-2\left(E_{n,l}V_0+m_0m_1+S_0m_0\right)B-D_1,\\D_1=\frac{(D+2l-1)(D+2l-3)C_1}{4}\\p_2=\left(V_0^2-S_0^2-2S_0m_1\right)B^2-D_2,\\D_2=\frac{(D+2l-1)(D+2l-3)C_2}{4}\end{array} \quad (10)$$

Equation (9) will be used solve using NU method in section 4.

PARAMETRIC NU METHOD

The NIkiforov-Uvarov method[7] and its parametric form[8] were proposed to solve the second order differential equation of the form

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \quad (11)$$

$$\frac{d^2\psi}{ds^2}+\frac{(\alpha_1-\alpha_2 s)}{s(1-\alpha_3 s)}\frac{d\psi}{ds}+\frac{1}{s^2(1-\alpha_3 s)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\psi(s)=0 \quad (12)$$

with appropriate co-ordinate transformation s = s(r), where σ(r) and σ(s) are polynomials at most a second degree and $\tilde{\tau}(s)$ is a first degree polynomial. The eigen function and the corresponding energy eigenvalues to the equation becomes

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)} \quad (13)$$

$$(\alpha_2-\alpha_3)n+\alpha_3n^2-(2n+1)\alpha_5+(2n+1)\left[\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right]+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0 \quad (14)$$

where

$$\begin{array}{l}\alpha_4=\frac{1}{2}(1-\alpha_1),\ \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\\\alpha_6=\alpha_5^2+\xi_1,\ \alpha_7=2\alpha_4\alpha_5-\xi_2,\\\alpha_8=\alpha_4^2+\xi_3,\ \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6\\\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}\\\alpha_{11}=\alpha_2-2\alpha_3+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)\\\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\ \boxed{\alpha_{13}=\alpha_3-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)}\end{array} \quad (15)$$

In the more special case of $\alpha_3 \to 0$,

$$\lim_{\alpha_3\to 0} P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}\left(1-2\alpha_3 s\right) = L_n^{\alpha_{10}-1}(\alpha_{11}s), \quad \textbf{(16a)}$$

$$\lim_{\alpha_3\to 0}\left(1-\alpha_3 s\right)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13}s}, \quad \textbf{(16b)}$$

and from Eq.(12), we find the wave function as,

$$\psi = s^{\alpha_{12}} e^{\alpha_{13}s} L_n^{\alpha_{10}-1}(\alpha_{11}s) \quad \textbf{(17)}$$

EXACT BOUND STATE SOLUTIONS OF KG EQUATION IN D-DIMENSIONS

In order to find the solution of Eq.(9), we use the transformation z = e^{-ar} and we obtain

$$\left\{\frac{d^2}{dr^2}+\frac{1}{z}\frac{d}{dr}+\frac{1}{z^2}\left(\frac{r_0^2 p_2}{\alpha^2}s^2+\frac{r_0^2 p_1}{\alpha^2}s+\frac{r_0^2 p_0}{\alpha^2}\right)\right\}U_{n,l}(z)=0. \quad \textbf{(18)}$$

Comparing Eq.(18) with Eq.(12), we obtain the following coefficients,

$$\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 0$$

$$\xi_1 = -\frac{r_0^2 p_2}{\alpha^2}, \xi_2 = \frac{r_0^2 p_1}{\alpha^2}, \xi_3 = -\frac{r_0^2 p_0}{\alpha^2} \quad \textbf{(19)}$$

The rest of the coefficients can be determine from Eq.(15) as

$$\begin{aligned}
&\alpha_4 = 0, \alpha_5 = 0, \alpha_6 = -\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right)\\
&\alpha_7 = \frac{r_0^2}{\alpha^2}\Big(2\left(V_0^2 - S_0^2 - m_1^2 - m_1 S_0\right)AB\\
&\quad -2\left(E_{n,l}V_0 + m_0 m_1 + S_0 m_0\right)B - D_1\Big),\\
&\alpha_8 = -\frac{r_0^2}{\alpha^2}\Big(E_{n,l}^2 - m_0^2 + \left(V_0^2 - S_0^2 - m_1^2 - 2S_0 m_1\right)A^2\\
&\quad -2\left(E_{n,l}V_0 + S_0 m_0 + m_0 m_1\right)A - D_0\Big),\\
&\alpha_9 = \frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right),\\
&\alpha_{10} = 1 + 2\sqrt{-\frac{r_0^2}{\alpha^2}\Big(E_{n,l}^2 - m_0^2 + \left(V_0^2 - S_0^2 - m_1^2 - 2S_0 m_1\right)A^2 -2\left(E_{n,l}V_0 + S_0 m_0 + m_0 m_1\right)A - D_0\Big)},\\
&\alpha_{11} = 2\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right)},\\
&\alpha_{12} = \sqrt{\frac{r_0^2}{\alpha^2}\Big(E_{n,l}^2 - m_0^2 + \left(V_0^2 - S_0^2 - m_1^2 - 2S_0 m_1\right)A^2 -2\left(E_{n,l}V_0 + S_0 m_0 + m_0 m_1\right)A - D_0\Big)},\\
&\alpha_{13} = -\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right)}.
\end{aligned} \quad \textbf{(20)}$$

Using Eqs.(14) and (20), we obtain the eigenvalues for the generalized exponential potential model as,

$$E_{n,l}^2 - m_0^2 = -\frac{\alpha^2}{4r_0^2}\left[\frac{\frac{r_0^2}{\alpha^2}\left[\left(2\left(V_0^2 - S_0^2 - m_1^2 - m_1 S_0\right)AB - 2\left(E_{n,l}V_0 + m_0 m_1 + S_0 m_0\right)B - D_1\right)\right]}{\left(n+\frac{1}{2}+\sqrt{\frac{r_0^2}{\alpha^2}\left[\left(-\left(V_0^2 - S_0^2\right)B^2 + 2\left(E_{n,l}V_0 + S_0 m_1\right)B + m_1^2 + D_2\right)\right]}\right)}\right]^2, \quad \textbf{(21)}$$

$$+2(E_{n,l}V_0 + S_0 m_0 + m_0 m_1)A - (V_0^2 - S_0^2 - m_0^2 - 2S_0 m_1)A^2 + D_0$$

and the corresponding wave function is obtain as,

$$U_{n,l}(r) = N_{n,l}\left(e^{-\alpha r}\right)^{\sqrt{\frac{r_0^2}{\alpha^2}\left(E_{n,l}^2 - m_0^2 + \left(V_0^2 - S_0^2 - m_1^2 - 2S_0 S_1\right)A^2 - 2\left(E_{n,l}V_0 + S_0 m_0 + m_0 m_1\right)A - D_0\right)}}$$

$$\left(e^{-\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right)}e^{-\alpha r}}\right)$$

$$L_n^{2\sqrt{-\frac{r_0^2}{\alpha^2}\left(E_{n,l}^2 - m_0^2 + \left(V_0^2 - S_0^2 - m_1^2 - 2S_0 S_1\right)A^2 - 2\left(E_{n,l}V_0 + S_0 m_0 + m_0 m_1\right)A - D_0\right)}} \quad \textbf{(22)}$$

$$\left(2\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right)B^2 + m_1^2 - D_2\right)}e^{-\alpha r}\right)$$

where $N_{n,l}$ is the normalization constant and $L_n^k(r)$ is the Laguerre polynomial.

DISCUSSIONS

In this section, we discuss some special cases. First case, when we set A = 0, B = -1, the generalized exponential potential model reduces to the exponential potential field reported by Hassanabadi et al[25] and one can obtain from Eqs.(21) and (22) the energy eigenvalues and the wave function for exponential potential as[25]

$$E_{n,l}^2 - m_0^2 = -\frac{\alpha^2}{4r_0^2}\left[\frac{\frac{r_0^2}{\alpha^2}\left[\left(2\left(E_{n,l}V_0 + m_0 m_1 + S_0 m_0\right) - D_1\right)\right]}{\left(n+\frac{1}{2}+\sqrt{\frac{r_0^2}{\alpha^2}\left[\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right) + m_1^2 - D_2\right)\right]}\right)}\right]^2 + D_0 \quad \textbf{(23)}$$

$$U_{n,l}(r) = N_{n,l}\left(e^{-\alpha r}\right)^{\sqrt{\frac{r_0^2}{\alpha^2}\left(-E_{n,l}^2 + m_0^2 + D_0\right)}}\left(e^{-\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right) + m_1^2 - D_2\right)}e^{-\alpha r}}\right.$$

$$L_n^{2\sqrt{\frac{r_0^2}{\alpha^2}\left(-E_{n,l}^2 + m_0^2 + D_0\right)}}\left(2\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2 - 2S_0 m_1\right) + m_1^2 - D_2\right)}e^{-\alpha r}\right), \quad \textbf{(24)}$$

Second case, when A = -1, B = 1, $m_1 = 0$, $\alpha \to 2\alpha$, the generalized exponential reduces to the Morse potential and its energy eigenvalues and the wave functions can be found from Eqs.(22) and (23) as[21],

$$E_{n,l}^2 - m_0^2 = -\frac{\alpha^2}{4r_0^2}\left[\frac{-\frac{r_0^2}{\alpha^2}\left[\left(2\left(V_0^2 - S_0^2\right) + 2\left(E_{n,l}V_0 + S_0 m_0\right) + D_1\right)\right]}{\left(n+\frac{1}{2}+\sqrt{\frac{r_0^2}{\alpha^2}\left[\left(\left(V_0^2 - S_0^2\right) - D_2\right)\right]}\right)}\right]^2 \quad \textbf{(25)}$$

$$-2(E_{n,l}V_0 + S_0 m_0) - (V_0^2 - S_0^2 - m_0^2) + D_0$$

$$U_{n,l}(r) = N_{n,l}\left(e^{-\alpha r}\right)^{\sqrt{\frac{r_0^2}{\alpha^2}\left(-E_{n,l}^2 + m_0^2 + D_0\right)}}\left(e^{-\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2\right) - D_2\right)}e^{-\alpha r}}\right.$$

$$L_n^{2\sqrt{\frac{r_0^2}{\alpha^2}\left(-E_{n,l}^2 + m_0^2 + D_0\right)}}\left(2\sqrt{\frac{r_0^2}{\alpha^2}\left(\left(V_0^2 - S_0^2\right) - D_2\right)}e^{-\alpha r}\right), \quad \textbf{(26)}$$

CONCLUSIONS

In this paper, we have obtained the approximate analytical solutions of the D-dimensional KG equation for generalized exponential-type potential model using the NU method. We have calculated the energy eigenvalues and the corresponding wave function expressed in terms of the Laguerre polynomials. The results can be useful in many branches of physics such as nuclear and particle physics[28], chemical physics[29], solid state physics[30] and molecular physics[31]. Finally, the limiting cases of this potential reduces to the exponential-type potential and Morse potential reported by Hassanabadi et al[25] and Hamzavi et al[21] respectively.

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