Regular Ternary Semigroups

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Abstract

Intriguing properties of regular ternary semigroups and completely regular ternary semigroups were discussed in the article.

Keywords: Regular ternary semigroup; Completely regular ternary semigroup

Introduction


Definition 1

An element $x$ in a ternary semigroup $T$ is said to be a regular if $\exists$ an element $a \in T$ s.t. $xax=x$ [2].

A ternary semigroup is said to be regular if all of its elements are regular.

Theorem 1

The following conditions in a ternary semigroup $T$ are equivalent:

(i) $T$ is regular.
(ii) For any right ideal $R$, lateral ideal $M$ and left ideal $L$ of $T$, $RML=R \cap M \cap L$.

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(iii) For \( x, y, z \in T \), \( \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l = \langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l \).

(iv) For \( x \in T \), \( \langle x \rangle_r \langle x \rangle_m \langle x \rangle_l = \langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l \).

**Proof**

(i) \( \Rightarrow \) (ii) Suppose \( T \) is a regular ternary semigroup. Let \( R, M \) and \( L \) be a right ideal, a lateral ideal and a left ideal of \( T \).

Then clearly, \( RML \subseteq R \cap M \cap L \). Now for \( x \in R \cap M \cap L \), we have \( x = axa \) for some \( a \in T \). This implies that \( x = axa = (axa)(axa)(axa) \in RML \).

Thus, we have \( R \cap M \cap L \subseteq RML \). So we find that \( RML = R \cap M \cap L \).

Clearly, (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv).

It remains to show that (iv) \( \Rightarrow \) (i).

Let \( x \in T \). Clearly, \( x \in \langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l = \langle x \rangle_r \langle x \rangle_m \langle x \rangle_l \).

Then we have, \( x = (xtT \cup nx)(TxT \cup TXT \cup nx)(TXT \cup nx) \subseteq xTx \).

So we find that \( x \in xTa \) and hence there exists an elements \( a \in T \) such that \( x = axa \). This implies that \( x \) is regular and hence \( T \) is regular.

We note that every left and right ideal of a regular ternary semigroup may not be a regular ternary semigroup.

However, for a lateral ideal of a regular ternary semigroup, we have the following result:

**Lemma**

Every lateral ideal of a regular ternary semigroup \( T \) is a regular ternary semigroup.

**Proof**

Let \( L \) be a lateral ideal of regular ternary semigroup \( T \). Then for each \( x \in L \) there exists \( a \in T \) such that \( x = axa \). Now \( x = axa = xaxax = x(axa)x = xpx \) where \( p = axa \in L \). This implies that \( L \) is a regular ternary semigroup.

**Definition 2**

An ideal \( A \) of a ternary semigroup \( T \) is said to be a regular ideal if \( A \cup RML = R \cap M \cap L \) for any right ideal \( R \supseteq A \), lateral ideal \( M \supseteq A \) and left ideal \( L \supseteq A \).

**Remark 1**

From Definition 2, it follows that \( T \) is always a regular ideal and any ideal that contains a regular ideal is also a regular ideal.

Now if for any right ideal \( R \), lateral ideal \( M \) and left ideal \( L \); \( RML \) contains a regular ideal, then \( RML = R \cap M \cap L \).

**Proposition**

A ternary semigroup \( T \) is a regular ternary semigroup if and only if \( \{0\} \) is a regular ideal of \( T \).
Proof
Let \( P \) be the nuclear ideal of a ternary semigroup \( T \), i.e., the intersection of all non-zero ideals of \( T \), \( P \) is the intersection of all non-zero right ideals of \( T \), \( P_m \) is the intersection of all non-zero lateral ideals of \( T \) and \( P_l \) is the intersection of all non-zero left ideals of \( T \). Now if \( P = \{0\} \), then clearly \( P = P_r = P_m = P_l \).

Theorem 2
Let \( T \) be a ternary semigroup and \( P = P_r = P_m = P_l \). Then \( T \) is a regular ternary semigroup if and only if \( P \) is a regular ideal of \( T \).

Proof
If \( P = P_r = P_m = P_l = \{0\} \), then proof follows from proposition. So we suppose that, \( P = P_r = P_m = P_l \neq \{0\} \). Let \( T \) be a regular ternary semigroup. Then from proposition, it follows that \( \{0\} \) is a regular ideal of \( T \). Now, \( \{0\} \subseteq P = P_r = P_m = P_l \) implies that \( P \) is a regular ideal of \( T \), by using Remark 1.

Conversely, let \( P \) be a regular ideal of \( T \). Then \( P \cup RML = R \cap M \cap L \) for any right ideal \( R \supseteq P \), lateral ideal \( M \supseteq P \) and left ideal \( L \supseteq P \) of \( T \). Since \( PPP \) is a right ideal of \( T \) and \( P = P_r \), we have \( P = P_r \subseteq PPP \subseteq RML \).

Consequently, \( P \cup RML = RML \). So \( RML = R \cap M \cap L \) and hence from Theorem 2, it follows that \( T \) is a regular ternary semigroup.

Corollary 1
Let \( T \) be a ternary semigroup and \( P = P_r = P_m = P_l \). Then \( T \) is a regular ternary semigroup if and only if every ideal of \( T \) is regular.

Proof
Suppose \( T \) is a regular ternary semigroup. Then from Theorem 2, it follows that \( P \) is a regular ideal of \( T \). Now \( P = P_r = P_m = P_l \) implies that every non-zero ideal of \( T \) contains the regular ideal \( P \) of \( T \). Consequently, by using Remark 1, we find that every ideal of \( T \) is regular.

Conversely, if every ideal of \( T \) is regular, then \( P \) is a regular ideal of \( T \) and hence from Theorem 2, it follows that \( T \) is a regular ternary semigroup.

Theorem 3
The following conditions in a ternary semigroup \( T \) are equivalent:
(i) \( A \) is a regular ideal of \( T \).
(ii) For \( x, y, z \in T \), \( A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l = A \cup (\langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l) \).
(iii) For \( x \in T \), \( A \cup \langle x \rangle_r \langle x \rangle_m \langle x \rangle_l = A \cup (\langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l) \).
(iv) For each \( x \in T \setminus A = A', x = \{a\} \cup \bigcup_{i=1}^{n}xp_i,xq_i,x \cup \bigcup_{i=1}^{n}xr_is_i,ux_i,v_i,x \) for some \( a \in A \) and \( p_i, q_i, r_i, s_i, u_i, v_i \in T \).
Proof

(i) \(\Rightarrow\) (ii) Suppose \(A\) is a regular ideal of \(T\). We note that for \(x, y, z \in T\),

\[ A \subseteq (A \cup \{x\}_r), (A \cup \{y\}_m), (A \cup \{z\}_l). \]

Now \(A \cup \{x\} \cap \{y\}_m \cap \{z\} \subseteq (A \cup \{x\}) \cap (A \cup \{y\}_m) \cap (A \cup \{z\}) = A \cup (A \cup \{x\}) \cup (A \cup \{y\}_m) \cup (A \cup \{z\})\) (since \(A\) is regular).

\[ \subseteq A \cup AAA \cup A\{y\}_m A \cup A\{y\}_m \{z\} \cup AA\{z\} \cup \{x\}, AA \cup \{x\}, A\{z\} \cup \{x\}, \{y\}_m \cup A \cup \{x\} \cup \{y\}_m \{z\} \]

\[ \subseteq A \cup \{x\} \cup \{y\}_m \{z\}. \]

Again \(\{x\} \cup \{y\}_m \{z\} \subseteq \{x\} \cap \{y\}_m \{z\}\) implies that \(A \cup \{x\} \cup \{y\}_m \{z\} \subseteq A \cup \{x\} \cap \{y\}_m \{z\}\).

So we find that \(A \cup \{x\} \cup \{y\}_m \{z\} = A \cup (\{x\} \cap \{y\}_m \{z\})\).

(ii) \(\Rightarrow\) (iii) Put \(y = x\) in (ii) we get (iii).

(iii) \(\Rightarrow\) (iv) We first note that \(\{A \cup \{x\}_r\} = A \cup \{x\} = A \cup \{x\} \cap T \cap T = A \cup \{x\}_r\).

\[ = A \cup (xTT \cup nx)TT = A \cup xTTTT \cup nxTT = A \cup \{xTT\}_r = A \cup xTT \]

Similarly we have, \(\{A \cup \{x\}_m\} = A \cup TxT \cup TTxTT\) and \(\{A \cup \{x\}_l\} = A \cup TTa\).

Now \(\{x\} \cap \{x\}_m \cap \{x\}_l \subseteq \{A \cup \{x\}_r\} \cap \{A \cup \{x\}_m\} \cap \{A \cup \{x\}_l\}\)

\[ \subseteq A \cap (\{A \cup \{x\}_r\} \cap \{A \cup \{x\}_m\} \cap \{A \cup \{x\}_l\}) \]

\[ = A \cup (\{A \cup \{x\}_r\} \cap \{A \cup \{x\}_m\} \cap \{A \cup \{x\}_l\}) \]

\[ = A \cup (A \cup xTT)(A \cup TxT \cup TTxTT)(A \cup TTa) \]

\[ \subseteq A \cup (xTTx \cup TTxTTx) \]

Since, \(x \in \{x\} \cap \{x\}_m \cap \{x\}_l\) there exists \(a \in A\) and \(p_i, q_i, r_i, s_i, u_i, v_i \in T\) such that

\[ x = \{a\} \cup \{x\} \cup \{x\}_m \cup \{x\}_l \]

(iv) \(\Rightarrow\) (i) Let \(R, M\) and \(L\) be any right, lateral and left ideal of \(T\) respectively such that \(R, M, L \supseteq A\). Then clearly, \(A \cup RML \subseteq R \cap M \cap L\). Again, let \(x \in R \cap M \cap L\). Then by using condition (iv), we have

\[ x = \{a\} \cup \{x\} \cup \{x\}_m \cup \{x\}_l \]

for some \(a \in A\) and \(p_i, q_i, r_i, s_i, u_i, v_i \in T\). Since

\[ \{x\}_m \cup \{x\}_l \subseteq RML, \]

\(x \in A \cup RML\) and hence \(R \cap M \cap L \subseteq A \cup RML\). Thus \(A \cup RML = R \cap M \cap L\). Consequently, \(A\) is a regular ideal.
Theorem 4
Let $A$ be a regular ideal of a ternary semigroup $T$. For any right ideal $R$, lateral ideal $M$ and left ideal $L$ of $T$, if $RML \subseteq A$ then $R \cap M \cap L \subseteq A$.

Proof
Suppose for any right ideal $R$, lateral ideal $M$ and left ideal $L$ of $T$, $RML \subseteq A$, where $A$ is a regular ideal of $T$. Then

$$A \subseteq (A \cup R), (A \cup M), (A \cup L).$$

Now $R \cap M \cap L \subseteq (A \cup R) \cap (A \cup M) \cap (A \cup L)$

$$= A \cup ((A \cup R)(A \cup M)(A \cup L)) \quad \text{[Since } A \text{ is regular]}$$

$$\subseteq A \cup AAA \cup AAL \cup AMA \cup AML \cup RAA \cup RAL \cup RMA \cup RML$$

$$\subseteq A.$$

From Theorem 4, we have the following results:

Corollary 2
A regular and strongly irreducible ideal of a ternary semigroup $T$ is a prime ideal of $T$.

Corollary 3
Every regular ideal of a ternary semigroup $T$ is a semi prime ideal of $T$.

Theorem 5
A ternary semigroup $T$ is regular if and only if every ideal of $T$ is idempotent.

Proof
Let $T$ be a regular ternary semigroup and $A$ be any ideal of $T$. Then $A^3 = AAA \subseteq TTA \subseteq A$. Let $x \in A$. Then there exists $a \in T$ such that $x = axa = axa$. Since $A$ is an ideal and $x \in A$, $axa \in A$. Thus $x = axa = axax \in A^3$.

Consequently, $A \subseteq A^3$ and hence $A^3 = AAA = A$ i.e., $A$ is idempotent.

Conversely, suppose that every ideal of $T$ is idempotent. Let $P$, $Q$ and $R$ be three ideals of $T$. Then $PQR \subseteq PTT \subseteq P$, $PQR \subseteq TQT \subseteq Q$ and $PQR \subseteq TTR \subseteq R$. This implies that $PQR \subseteq P \cap Q \cap R$. Also, $(P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R) \subseteq PQR$. Again, since $(P \cap Q \cap R)$ is an ideal of $T$, $(P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R) = P \cap Q \cap R$. Thus $P \cap Q \cap R \subseteq PQR$ and hence $P \cap Q \cap R = PQR$. Therefore, by Theorem 2, $T$ is a regular ternary semigroup.

Theorem 6
A ternary semigroup $T$ is left (resp. right) regular if and only if every left (resp. right) ideal of $T$ is completely semiprime.
Proof
Let \( T \) be a left regular ternary semigroup and \( L \) be any left ideal of \( T \). Suppose \( a^3 = aaa \in L \) for \( a \in T \). Since \( T \) is left regular, there exists an element \( x \in T \) such that \( a = xaa = x(xaa)a = xx(aaa) \in TTL \subseteq L \). Thus \( L \) is completely semiprime.

Conversely, suppose that every left ideal of \( T \) is completely semiprime. Now for any \( a \in T \), \( Ta \) is a left ideal of \( T \). Then by hypothesis, \( Ta \) is a completely semiprime ideal of \( T \). Now \( a^3 = aha \in Ta \). Since \( Ta \) is completely semiprime, it follows that \( a \in Ta \). So there exists an element \( x \in T \) such that \( a = xaa \). Consequently, \( a \) is left regular. Since \( a \) is arbitrary, it follows that \( T \) is left regular.

Equivalently, we can prove the Theorem for right regularity.

**Completely Regular Ternary Semigroup**

**Definition 3**
A pair \((p, q)\) of elements in a ternary semigroup \( T \) is known as an idempotent pair if \( pq(pqx) = pqx \) and \( (xpq)pq = xpq \) for all \( x \in T \) [3].

**Definition 4**
Two idempotent pairs \((p, q)\) and \((r, s)\) of a ternary semigroup \( T \) are known as an equivalent, if \( pqx = rsx \) and \( xpq = xrs \) for all \( x \in T \) [3].

In notation we write \((p, q) \sim (r, s)\).

**Definition 5**
An element \( x \) of a ternary semigroup \( T \) is said to be completely regular if \( \exists \) an element \( a \in T \ \exists \ xax = x \) and the idempotent pairs \((a, x)\) and \((x, a)\) are equivalent.

If all the elements of \( T \) are completely regular, then \( T \) is called completely regular [3].

**Definition 6**
An element \( x \) of a ternary semigroup \( T \) is known as a left regular if \( \exists \) an element \( a \in T \ \exists \ axa = x \)

**Definition 7**
An element \( x \) of a ternary semigroup \( T \) is said to be right regular if \( \exists \) an element \( a \in T \ \exists \ xxa = x \)

**Theorem 7**
A ternary semigroup \( T \) is completely regular then \( T \) is left and right regular. [i.e., \( x \in x^2T \cap Tx^2 \) for all \( x \in T \).]
Proof
Suppose T is a completely regular ternary semigroup. Let \( x \in T \). Then there exists an element \( a \in T \) such that \( xax = x \) and the idempotent pairs \((x, a)\) and \((a, x)\) are equivalent i.e., \( xab = axb \) and \( bxa = bax \) for all \( b \in T \). Now in particular, putting \( b = x \) we find that \( xax = axx \) and \( xaa = xax \). This implies that \( x \in xxT \) and \( x \in Txx \). Hence T is left and right regular.

Theorem 8
A ternary semigroup T is left and right regular then \( x \in x^2Tx^2 \) for all \( x \in T \).

Proof
Suppose that T is both left and right regular. Let \( x \in T \). Then \( \exists p, q \in T \) such that \( x = xxp \) and \( x = qxq \). This implies that \( xpz = qxz = qxz \) for all \( z \in T \).

Now \( x = xxp = x(xxp)p = x^2(xxp)p = x^2q(xx)p = x^2q^2x = x^2q^2x^2 = x^2q^3x \in x^2Tx^2 \). Hence \( x \in x^2Tx^2 \) for all \( x \in T \).

Theorem 9
If T is ternary semigroup \( x \in x^2Tx^2 \) for all \( x \in T \) then T is completely regular.

Proof
Suppose \( x \in x^2Tx^2 \) for all \( x \in T \). Then \( \exists a \in T \) such that \( x = x^2ax^2 \)

Now \( x = x^2ax^2 = x(xax)x = xba \), where \( b = xax \in T \). This implies that T is regular. Also \( xbc = x(xax)c = x^2ax^2c \) and \( bxc = (xax)xc = x^2ax^2c \) for all \( c \in T \). This shows that the idempotent pairs \((x, b)\) and \((b, x)\) are equivalent.

Consequently, T is a completely regular ternary semigroup.

Definition 8
A sub semigroup S of a ternary semigroup T is said to be a bi-ideal of T if \( STST \subseteq S \).

Theorem 10
A ternary semigroup T is completely regular ternary semigroup if and only if every bi-ideal of T is completely semiprime.

Proof
Let T be a completely regular ternary semigroup. Let P be any bi-ideal of T. Let \( p^3 \in P \) for \( p \in T \). Since T is completely regular, from Theorem 10, it follows that \( p \in p^2Tp^2 \). This implies that there exists \( x \in T \) such that \( p = p^2xp^2 = p(p^2xp^2)x(p^2xp^2)p = p^3(xp^2x)p(p^2xp^2)xp^3 = p^3(xp^2x)p^3(xp^2x)p^3 \in PTTP \subseteq P \). This shows that P is completely semiprime.
Conversely, assume that every bi-ideal of T is completely semiprime. Since every left and right ideal of a ternary semigroup T is a bi-ideal of T, it follows that every left and right ideal of T is completely semiprime. Consequently, we have from Theorem 6 that T is both left and right regular. Now by using Theorem 9, we find that T is a completely regular ternary semigroup.

**Theorem 11**

If T is a completely regular ternary semigroup, then every bi-ideal of T is idempotent.

**Proof**

Let T be a completely regular ternary semigroup and P be a bi-ideal of T. Clearly T is a completely regular ternary semigroup. Let \( p \in P \). Then there exists \( x \in T \) such that \( p = pxp \). This implies that \( p \in PTP \) and hence \( P \subseteq PTP \). Also \( PTP \subseteq PTP \subseteq P \). Thus we find that \( P = PTP \). Again, we have from Theorem 11 that \( p \in P^2TP \subseteq P^2TP \). This implies that \( p \subseteq P^2TP = P(PTP) = PPP \subseteq P \). Hence \( P^3 \subseteq P \). Therefore every bi-ideal of P is idempotent.

**Conclusion**

Ternary structures and their speculation, the purported \( n \)-ary structures bring certain expectations up in perspective of their conceivable applications in organic chemistry.

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