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PROPERTIES OF HEAVISIDE'S UNIT STEP FUNCTION BASED ON LAPLACE – STIELTJES TRANSFORM

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ABSTRACT

Laplace-Stieltjes transform is one of the flourishing field of active research due to its wide range of applications. Purpose of this paper is to prove some properties of unit step function with the help of Laplace - Stieltjes transform. It seldom matters what value is used for U (0), since U is mostly used as a distribution. Some common choices are the function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely. It is also used in structural mechanics together with the Dirac Delta function to describe different types of structural loads.

Key words: Laplace Transform, Laplace-stieltjes transform, Testing function space, Unit step function.

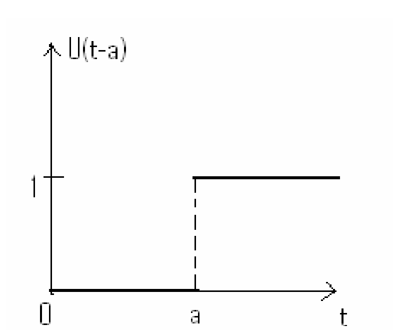
INTRODUCTION

The unit step function also called Heaviside's unit step function U(t), is defined as -

$$U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

The unit step function is a curve which has value zero at all points to the left of the origin and is equal to one on the right of the origin. The displaced unit step function U(t-a) is defined as -

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



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Where a ≥ 0. The notation and terminology will follow that of Zemanian[2].The conventional LS transform of a complex valued smooth function φ (t, x) is defined by the convergent integral.

$$F(s, y) = LS \{f(t, x)\} = \int_{00}^{\infty\infty} f(t,x)e^{-s\tau}(x+y)^{-p}\,dtdx$$

Testing function space $LS_{a,\alpha}$ (Ω)

A function ψ(*t,x*) defined on 0 < t < ∞, 0 < x < ∞ is said to be a member of $LS_{a,\alpha}$ if ψ(*t,x*) is smooth and for each non negative integer l,q

$$\gamma_{a,k,l,q}(\psi) = \underset{0<\tau,\, x<\infty}{Sup} [(\ |e^{a\tau})](1+x)^k\, D_\tau^l (x\,D_x)^q\, \psi(t,x)|$$
$$\le C_q\, A^k\, k^{ka}, \qquad k = 0,1,2$$

Where the constants A and C_q depend on the testing ψ. For k = 0, we get k^{ka} =1. The topology of the space $LS_{a,\alpha}$ is the space generated by the countable multinorm, S = $\bigcup_{l,k=0}^{\infty}\{\gamma_{a,k,l,q}\}$. With this topology $LS_{a,\alpha}$ is a countably multinorm space. A sequence $\{\psi_v\}$ is said to converge in $LS_{a,\alpha}$ to ψ if for each non negative integer *l,q* we have, $\gamma_{a,k,l,q}(\psi_v - \psi) \to 0$ as $v \to \infty$. We define distribution LS transform of any function f in dual space $LS_{a,\alpha}$ i.e. $LS^*_{a,\alpha}$ by LS {*f*(*t,x*)} = F (S,Y) = {*f*(*t,x*), $e^{-s\tau}(x+y)^{-p}$}. For complex parameters S and Y. The R.H.S. has sence for $f \in LS^*_{a,\alpha}$ and $e^{-s\tau}(x+y)^{-p} \in LS_{a,\alpha}$.

Properties of heaviside's unit step function on ls transform

(a) (*f* + *g*, φ) = *f*, φ) + (*g*, φ)

(b) (α*f* , φ) = (*f*, φ) + (*g*, α φ)

Proof : (a) (*f* + *g*, φ) = $\left(f+g, \dfrac{e^{-s\tau}}{(s+t)^p}\right)$

$$= \int_{-\infty}^{\infty} f(t)\frac{e^{-s\tau}}{(s+t)^p}\,dt + \int_{-\infty}^{\infty} g(t)\frac{e^{-s\tau}}{(s+t)^p}\,dt$$

= (f, φ) + (g, φ)

Proof : (b) This also we can prove.

2. {*f*(*t* – α), φ(*t*)} = {*f*(*t*), φ(*t* + α)}

Proof : {*f*(*t* – α), φ(*t*)} = $\displaystyle\int_{-\infty}^{\infty} f(t-\alpha)\frac{e^{-s\tau}}{(s+t)^p}\,dt$

$$= \int_{-\infty}^{\infty} f(x)\frac{e^{-s(x+\alpha)}}{[s+(x+\alpha)]^p}\,dt \qquad (\text{put } t - \alpha = x)$$

= {f(t), φ (*t* + α)}

3. $\{f(-t), \varphi(t)\} = \{f(t), \varphi(-t)\}$

Proof : $\{f(-t), \varphi(-t)\} = \int_{-\infty}^{\infty} f(-t)\frac{e^{-s\tau}}{(s+t)^p}\,dt$

$$= \int_{-\infty}^{\infty} f(x)\frac{e^{-s(-x)}}{[s+(-x)]^p}(-)\,dt \qquad \text{(put } -t = x\text{)}$$

$$= \int_{-\infty}^{\infty} f(x)\frac{e^{-s(-x)}}{[s+(-x)]^p}\,dt$$

$$= \left\{\{f(t),\ \frac{e^{-s(-\tau)}}{[s+(-t)]^p}\right\}$$

$$= \{f(t), \varphi(-t)\}$$

4. $\{f(a\,t), \varphi(t)\} = \left\{f(t), \frac{1}{a}\boldsymbol{\varphi}\left(\frac{t}{a}\right)\right\}$

Proof : $\{f(a\,t), \varphi(t)\} = \int_{-\infty}^{\infty} f(at)\frac{e^{-s\tau}}{(s+t)^p}\,dt$

$$= \frac{\int_{-\infty}^{\infty} f(x)\frac{e^{-s\tau\{\frac{x}{a}\}}}{[s+\{\frac{x}{a}\}]^p}\,dt}{a} \qquad \text{(put at = x)}$$

$$= \left\{f(t), \frac{1}{a}\boldsymbol{\varphi}\left(\frac{t}{a}\right)\right\}$$

(a) $\{f'(t), \varphi(t)\} = \{f(t), -\varphi'(t)\}$

(b) $\{f'(t), \varphi(t)\} = \left\{f(t), \left[s+\frac{p}{(s+t)}\right]\boldsymbol{\varphi}(t)\right\}$

Proof : (a) $\{f'(t), \varphi(t)\} = \int_{-\infty}^{\infty} f'(t)\frac{e^{-s\tau}}{(s+t)^p}\,dt$

$$= \int_{-\infty}^{\infty} f(t)\left[-\frac{d}{dt}\left(\frac{e^{-s\tau}}{(s+t)^p}\right)\right]dt$$

$$= \{f(t), -\varphi'(t)\}$$

Proof : (b) $\{f'(t), \varphi(t)\} = \int_{-\infty}^{\infty} f'(t)\frac{e^{-s\tau}}{(s+t)^p}\,dt$

$$= \int_{-\infty}^{\infty} \frac{(s+t)^p e^{-s\tau}(-s) - e^{-s\tau}(s+t)^{p-1}p}{(s+t)^{2p}}\,f(t)dt$$

$$= \int_{-\infty}^{\infty} f(t)\,\frac{e^{-s\tau}}{(s+t)^p}\left[s+\frac{p}{(s+t)}\right]dt$$

$$= \left\{f(t), \left[s+\frac{p}{(s+t)}\right]\boldsymbol{\varphi}(t)\right\}$$

6. $\left\{\frac{d}{dt} f(a\,t),\ \boldsymbol{\varphi}(t)\right\} = \left\{f(t), \frac{-d}{dt}\boldsymbol{\varphi}\left(\frac{t}{a}\right)\right\}$

Proof : $\left\{\frac{d}{dt} f(a\,t),\ \boldsymbol{\varphi}(t)\right\} = \int_{-\infty}^{\infty} \frac{d}{dt} f(a\,t)\,\frac{e^{-s\tau}}{(s+t)^p}\,dt$

Now on integrating by parts and putting a $t = x$, we get -

$$= -\int_{-\infty}^{\infty} \frac{d}{dt} f(a\,t)\,\frac{e^{-\frac{sx}{a}}}{\left(s+\left(\frac{x}{a}\right)\right)^p}\,f(x)\,dt$$

$$= -\int_{-\infty}^{\infty} \frac{d}{dt}\,\frac{e^{-\frac{s\tau}{a}}}{\left(s+\left(\frac{t}{a}\right)\right)^p}\,f(t)\,dt$$

$$= \left\{f(t), \frac{-d}{dt}\boldsymbol{\varphi}\left(\frac{t}{a}\right)\right\}$$

If f is a distribution and ψ is a function which is infinitely smooth then –

$$\left\{\frac{\partial}{\partial t(\boldsymbol{\psi} f)}, \boldsymbol{\varphi}\right\} = \left\{\boldsymbol{\psi}\frac{\partial f}{\partial t}, \boldsymbol{\varphi}\right\} + \left\{f\,\frac{\partial \boldsymbol{\psi}}{\partial t}, \boldsymbol{\varphi}\right\}$$

Proof : $\left\{\frac{\partial}{\partial t\,(\boldsymbol{\psi}\ f)}, \boldsymbol{\varphi}\right\} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\psi\ f)\,(t)\,\boldsymbol{\varphi}(t)\,dt$

$$= \int_{-\infty}^{\infty} \frac{(\boldsymbol{\psi} f)(t)(-\partial)}{\partial t}\,\boldsymbol{\varphi}(t)\,dt \qquad \text{(using prop. 5(a).)}$$

$$= \int_{-\infty}^{\infty} f(t)\,(-\boldsymbol{\psi})\,\frac{\partial \boldsymbol{\varphi}}{\partial t}\,dt \qquad \text{(using prop. 1(b).)}$$

$$= \int_{-\infty}^{\infty} [f(t)\left[\frac{-\partial}{\partial t}](\boldsymbol{\psi}\,\boldsymbol{\varphi}) + \boldsymbol{\varphi}\frac{\partial \boldsymbol{\psi}}{\partial t}\right]dt$$

$$= \left\{f(t), \frac{-\partial}{\partial t}\boldsymbol{\psi}\,\boldsymbol{\varphi}\right\} + \left\{f(t), \boldsymbol{\varphi}\frac{\partial \boldsymbol{\psi}}{\partial t}\right\}$$

$$= \left\{\frac{\partial f}{\partial t}, \boldsymbol{\psi}\, \boldsymbol{\varphi}\right\} + \left\{ f\,\frac{\partial \boldsymbol{\psi}}{\partial t}, \boldsymbol{\varphi}\right\}$$

$$= \left\{\frac{\partial f}{\partial t}, \boldsymbol{\varphi}\right\} + \left\{ f\,\frac{\partial \boldsymbol{\psi}}{\partial t}, \boldsymbol{\varphi}\right\}$$

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