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# Pati-Salam Model in Curved Space-Time and Sheaf Quantization

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## Abstract

**This note aims to give a self-contain and detail explaination about U(k′) × U(k) Pati-Salam model in curved space-time which derived from (1 + n)-dimensional square root Lorentz manifold by self-parallel transportation principle. The concepts fundation of manifold from view point of category theory, fiber bundle and sheaf theories are reviewed. There are extra U(k′) × U(k)-principal bundle and U(k)-associated bundle than (1 + n)-dimensional Lorentz manifold. The conservative currents on square root Lorentz manifold is discussed preliminary. A detail proof of relation from sheaf quantization to path integral quantization is given.**

***Keywords: Sheaf quantization; Pati-Salam model; Yang-Mills theory; Quantum field theory in curved space-time***

## Introduction

Square root metric manifold has extra U(k′) ×U(k) principal bundle and U(k)-associated bundle than usual Lorentz manifold. These extra bundles gives us opportunity to con-struct Yang-Mills theory in curved space-time [1], especially the Pati-Salam type Yang-Mills theory [2, 3] in curved space-time. Sheaf as a natural mathematic structure being found by mathematicians [4], for example, Jean Leray, long ago.

Sheaf theory has deep relation with fiber bundle theory [5] (Yang-Mills theory [6, 7]) and superposition principle. Sheaf can be derived from contravariant functor in category theory, the sheaf cohomology and spectral sequences is fascinating and useful. The micro support language of sheaf the-ory [4] from Mikio Sato might be popular in future mathematic-physicsts.

Sheaf as a basic language of topos from Grothendieck [8], “we cannot even define a scheme without using scheaves” [9]. Sheaf quantization might be a method to quantize quantum field theory in curved space-time which avoiding problem of infinities [1, 16-17].

The sheaf space is linear space and coherent with superposition principle, even the base manifold is curved. The sheaf quantization method is consistent with path integral quan-tization method.

In this paper, the section 2 gives us a prilimninary concepts introduction of category, functor; and the topological space, sheaf, manifold, bundle from the category point of view.

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The section 3 talk about Einstein-Cartan geometry of Lorentz and Riemann manifold. The section 4 is a brief introduction of generators of Clifford algebra. The section 5 describes the geometry framework of square root Lorentz manifold. Based on square root Lorentz manifold, the Pati-Salam model in curved space-time and Einstein-Cartan gravity are constructed. The section 6 discusses the formulation of sheaf quantization, and the relation between sheaf quantization and path integral.

## Category, functor, topological space, sheaf, manifold and fiber bundle

### Category

The category C consist of

a class ob(C) of objects, for example, a, b, c, d $\in$ ob(C)

$$\mathrm{hom}_C(a, b) \tag{2.1}$$

represent all morphisms from a to b in category C. For example,

$$f, g \in \mathrm{hom}(a, b),\ h \in \mathrm{hom}(b, c),\ i \in \mathrm{hom}(a, c). \tag{2.2}$$

Composition of morphisms is, for objects a, b, c $\in$ ob(C),

$$\mathrm{hom}_C(a, b) \times \mathrm{hom}_C(b, c) \to \mathrm{hom}_C(a, c). \tag{2.3}$$

The morphisms hom(C) in category C satisfy the axiom of associativity and iden-tity:

(Associativity axiom) if

$$f : a \to b,\ g : b \to c,\ h : c \to d,) \tag{2.4}$$

then

$$h(gf) = (hg)f. \tag{2.5}$$

(Identity axiom) For every object x, y $\in$ ob(C), there exists a morphism

$$1_x : x \to x. \tag{2.6}$$

For every morphism f $\in$ hom(C)

$$f : x \to y, \tag{2.7}$$

we have

$$1_x f = f = f1y. \tag{2.8}$$

### Functor

Functors are structure-preserving maps between categories. A covariant functor F from a category C to a category D is written

$$F : C \to D, \tag{2.9}$$

and the structure-preserving means

for object x $\in$ ob(C) and F (x) $\in$ ob(D) and morphisms f $\in$ hom(C)

$$f : x \to y,\ F(f) : F(x) \to F(y), \tag{2.10}$$

where

$$f \in \mathrm{hom}(C),\ F(f) \in \mathrm{hom}(D). \tag{2.11}$$

such that,

For every object x $\in$ ob(C),

$$F(1_x) = 1_{F(x)}; \tag{2.12}$$

for objects $x, y, z \in ob$(C), all morphisms in C

$$f : x \to y, \qquad g : y \to z, \tag{2.13}$$

the functor preserves the composition of morphisms

$$F(gf) = F(g)F(f). \tag{2.14}$$

A contravariant functor like structure-preserving covariant functor from categories C to D, but for morphism $f, g \in hom$(C)

$$f : x \to y, \quad \Rightarrow \quad F(f) : F(y) \to F(x), \tag{2.15}$$

$$F(gf) \quad = \quad F(f)F(g). \tag{2.16}$$

**Topological space**

The point $x_0 \in x$ and the neighborhood of $x_0$ (an open covering )

$$U_{x_0} = \{x | x \to x_0\} \tag{2.17}$$

can glun to topological space X (more pricisely, an open covering Hausdorff space X )

$$\mathbf{X} = \cup U_x, \tag{2.18}$$

where

$$x \to x_0 \tag{2.19}$$

is the direct limit from $x$ to $x_0$. For any point $x_0 \in x$, there is open covering partial ordered set on topological space X

$$U_{x_0} \subset U^1_{x_0} \subset U^2_{x_0} \subset \cdots \mathbf{X}. \tag{2.20}$$

**Category Viewing of Topological Space**

- The Topological space X is a category with objects

$$U_{x_0} \in ob(\mathbf{X}), \quad x_0 \in x. \tag{2.21}$$

  and morphisms

$$\subset, \cup, \cap \in hom(\mathbf{X}). \tag{2.22}$$

- The category Top with objects

$$\mathbf{X} \in ob(\mathbf{Top}). \tag{2.23}$$

  and morphisms

$$\textit{continuous map} \in hom(\text{Top}). \tag{2.24}$$

**Presheaf and Sheaf**

$F(U_{x0})$ is the presheaf on $U_{x0}$ which is isomorphic to Abel group A ($F(U_{x0})$ means all possible functions on neighborhood $U_{x0}$)

$$F: U_{x_0} \to F(U_{x_0}). \tag{2.25}$$

$F$ is a functor from neighborhood $U_{x0}$ to presheaf $F(U_{x0})$. From presheaf $F(U_{x0})$ to construct sheaf $F$(X) satisfy the locality axiom and gluing axiom.

- (Locality axiom) If $U_{x0}$ is an open covering of an open set X, and if sections

  $$s_x, t_x \in F(\mathbf{X}), \tag{2.26}$$

  such that for any $x0 \in$ X

  $$s|_{U_{x_0}} = t|_{U_{x_0}}, \tag{2.27}$$

  then

  $$s_x = t_x, \tag{2.28}$$

  where $s|U_{x0}$ is the section restricted to neighborhood of $x_0$.

- (Gluing axiom) If

  $$x0, x1 \in \text{X}, \tag{2.29}$$

  $U_{x0}$ and $U_{x1}$ are open covering of an open set X, and for sections

  $$s_{x0} \in F(U_{x0}), s_{x1} \in F(U_{x1}), \text{ the} \tag{2.30}$$

  sections agree on the overlap

  $$s_{x0}|U_{x0} \cap U_{x1} = s_{x1}|U_{x1} \cap U_{x0}, \tag{2.31}$$

  the presheaf gluing aixom (2.31) can be presented by commutative diagram

  $$\begin{array}{ccc} F(U_{x_0}) & \xrightarrow{\cap F(U_{x_1})} & F(U_{x_0} \cap U_{x_1}) \\ \Big| F & & \Big| F \\ U_{x_0} & \xrightarrow{\cap U_{x_1}} & U_{x_0} \cap U_{x_1} \end{array} \tag{2.32}$$

  then there is a global section

  $$s_x \in F(\mathbf{X}), \quad x \in \mathbf{X}, \tag{2.33}$$

  such that

  $$s_{x_0} = s_x|U_{x_0}. \text{ The} \tag{2.34}$$

  stalk of $x_0$ is the sheaf space restricted to $x_0$

$$F_{x_0} = F\,(\mathrm{X})|U_{x_0} = F\,(U_{x_0}\,)/\sim, \tag{2.35}$$

where $\sim$ is an equivalence relation from restriction.

## Manifold

For manifold, in the neighborhood $U_{x0}$ of $x_0$, there is coordinate

$$dx^{\mu}|_{x\to x_0}\,. \tag{2.36}$$

As an example of presheaf, the collection of all possible coordinates in the neighborhood $U_{x0}$ of $x0$ is a presheaf

$$d(U_{x0}\,) = \{dx^{\mu}|_{x\to x0}\,\}. \tag{2.37}$$

The presheaf $d(U_{x0}\,)$ can gluon to sheaf $d$(X) because

$$\begin{array}{ccc} d(U_{x_0}) & \xrightarrow{\cap d(U_{x_1})} & d(U_{x_0}\cap U_{x_1}) \\ \Big| d & & \Big| d \\ U_{x_0} & \xrightarrow{\cap U_{x_1}} & U_{x_0}\cap U_{x_1} \end{array} \tag{2.38}$$

.

$$d:\mathbf{X}\to d(\mathbf{X}). \tag{2.39}$$

where differential structure $d$(X) is collection of all possible globle coordinates on topo-logical space X, $d$ is one kind of functor $F$ , and $d$(X) is one kind of sheaf $F$ (X). Then we can see the manifold M is topological space X with differential structure $d$(X) ( the globle coordinates on $(1 + n)$-dimensional manifold M might not be parameteried by $R^{1+n}$)

$$\mathbf{M} = (\mathbf{X}, d(\mathbf{X})). \tag{2.40}$$

- We point out that the definition of manifold M = (X, $d$(X)) is equivalent with the definition in usual book with axioms of locally flatness and atlas compatibility.
- (Locally flatness axiom) The point $x0$ in $(1 + n)$-dimensional manifold, then the neighborhood $U_{x0}$ can isomorphic to $R^{1+n}$.
- (Atlas compatibility axiom) The points $x0$ and $x1$ in $(1 + n)$-dimensional manifold have neighborhood $U_{x0}$ and $U_{x1}$ with parametrization $\{x_0{}^{\mu},\ \mu = 0, 1, 2, \cdots, n\}$ and $\{x_1{}^{\mu},\ \mu = 0, 1, 2, \cdots, n\}$. Then, there are coordinates $\{dx_0{}^{\mu}$, $\mu = 0, 1, 2, \cdots$ , n and $\{dx_0{}^{\mu}$, $\mu = 0, 1, 2, \cdots , n\}$. For the overlap of the two neighborhood

$$U_{x0} \cap U_{x1}\,, \tag{2.41}$$

there is coordinate transformation

$$dx_0{}^{\mu} = \Lambda^{\mu}{}_{\nu}\,(x0)dx^{\nu}{}_1 = \Lambda^{\mu}{}_{\nu}\,(x1)dx^{\nu}{}_1\,, \qquad \Lambda^{\mu}{}_{\nu}\,(x0), \Lambda^{\mu}{}_{\nu}\,(x1) \in GL(1+n,\mathbb{R}), \tag{2.42}$$

where

$$\Lambda^{\mu}{}_{\nu}\,(x0) = \Lambda^{\mu}{}_{\nu}\,(x)|_{x\to x^0}\,. \tag{2.43}$$

- For any number of neighborhood, there is

$$dx_0^{\mu} = \Lambda^{\mu}{}_{\nu_1}(x_0)\Lambda^{\nu_1}{}_{\nu_2}(x_1)\cdots\Lambda^{\nu_{q-1}}{}_{\nu_q}(x_q)\quad dx_q^{\nu_q} = Hom\left(\Lambda^{\mu}{}_{\nu}(x_0), \Lambda^{\mu}{}_{\nu}(x_q)\right) dx_q^{\nu}, (2.44) \tag{2.44}$$

where the element in $Hom\left(\Lambda^{\mu}{}_{\nu}(x_0), \Lambda^{\mu}{}_{\nu}(x_q)\right)$ is path dependent and cotangent principal bundle section dependent element of linear transformation group $GL(n, \mathbb{R})$ valued. Then

$$x_0{}^{\mu} = Hom\ (\Lambda^{\mu}{}_{\nu}\ (x_0),\ \Lambda^{\mu}{}_{\nu}\ (x_q))\ x_q{}^{\nu} + C_q - C_0. \tag{2.45}$$

Which means the parameters in x0 and xq relies on the continues path

$$C : \tau \to \mathbf{M}, C(\tau^k) = x^k, k = 0, 1, 2, \cdots, n, \tag{2.46}$$

linear transformation

$$Hom\left(\Lambda^{\mu}{}_{\nu}(x_0), \Lambda^{\mu}{}_{\nu}(x_q)\right) \tag{2.47}$$

and the edge function $C_q - C_0$. The solutions of equation (2.44) just the sheaf space $d$(X) restricted on curve $C(\tau)$

$$d(\mathbf{X})|_{C(\tau)}. \tag{2.48}$$

From equation (2.45) we can see that the global coordinates in (1 + $n$)-dimensional manifold might not parameteried by $R^{1+n.}$

## Category Viewing of Manifold

- The manifold M is a category with objects

$$U_{x_0}, d(U_{x_0}) \in ob(\mathbf{M}), \tag{2.49}$$

and morphisms

$$\subset, \cup, \cap, d \in hom(\mathbf{M}). \tag{2.50}$$

- The category Man with objects

$$\mathbf{M} \in ob(\mathbf{Man}), \tag{2.51}$$

and morphisms

$$\text{continuous differentiable map} \in \text{hom(Man)}. \tag{2.52}$$

## Principal Bundle

The fiber $E(U_{x0})$ of the cotangent principal bundle $E$(M) on manifold M isomorphic to the freedom $G$ = $GL$(1+$n$, R) of coordinates can make transformation (left action) locally

$$E(U_{x0}) = \{\Lambda^{\mu}{}_{\nu}(x)|_{x\to x0} \mid dx^{\mu'}|_{x'\to x0} = \Lambda^{\mu}{}_{\nu}(x)dx^{\nu}|_{x\to x0}, \Lambda^{\mu}{}_{\nu}(x)|_{x\to x0} \in GL(1+n, \text{R})\}. \tag{2.53}$$

The cotangent principal $G$-bundle $E$(M) on manifold M is

$$E(\mathbf{M}) = \cup E(U_x), \qquad G = GL(1+n, \mathbb{R}), \tag{2.54}$$

so the cotangent principal bundle is a map $\pi$ from total space $E$ to base manifold M

$$\pi : E \to \mathbf{M}. \tag{2.55}$$

$$\begin{array}{ccccc} E(U_{x0}) & \xrightarrow{\cup E(U_x)} & E(U_{x_0} \cup U_x) & \xrightarrow{\cup} & E(\mathbf{M}) \\ \Big\downarrow \pi & & \Big\downarrow \pi & & \Big\downarrow \pi \\ U_{x_0} & \xrightarrow{\cup U_x} & U_{x_0} \cup U_x & \xrightarrow{\cup} & \mathbf{M} \end{array} \tag{2.56}$$

The inverse mapping of $\pi$ is a section of the sheaf $d$(M) and bundle $E$(M)

$$\pi^{-1} \in d(\mathbf{M}), \quad \pi^{-1} \subset E(\mathbf{M}). \tag{2.57}$$

Here for inverse mapping of cotangent principal bunlde, the meaning of $\pi^{-1}$ is one global coordinate of the manifold M. The contravariant functor $\hat{\pi}^{-1}$ of $\pi$ is the differential

structure sheaf of the manifold M (all possible global coordinates). Because we have the commutative diagram

$$\begin{array}{ccccc} d(U_{x_0}) & \xrightarrow{\cup d(U_x)} & d(U_{x_0} \cup U_x) & \xrightarrow{\cup} & d(\mathbf{M}) \\ \hat{\pi}^{-1} \Big| d & & \hat{\pi}^{-1} \Big| d & & \hat{\pi}^{-1} \Big| d \\ U_{x_0} & \xrightarrow{\cup U_x} & U_{x_0} \cup U_x & \xrightarrow{\cup} & \mathbf{M} \end{array} \tag{2.58}$$

then

$$\hat{\pi}^{-1} = d. \tag{2.59}$$

The tangent principal bundle $E^*$(M) is the dual bundle of cotangent principal bundle $E$(M)

$$\pi^* : E^* \to \text{M}. \tag{2.60}$$

The section $\pi^{*-1}$ in the neighborhood of $U_{x0}$ has the formula

$$\left.\frac{\partial}{\partial x^\mu}\right|_{x \to x_0} \tag{2.61}$$

and dual with coordinates

$$\left.\langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle\right|_{x \to x_0} = \delta^\mu_\nu. \tag{2.62}$$

The sheaf $\hat{\pi}^{*-1}$ is dual with $\hat{\pi}^{-1}$. The right action of element of $GL(1+n, \text{R})$ on tangent principal bundle is consistent with the definition of left action transformation on cotangent bundle

$$\left(\frac{\partial}{\partial x^\mu}\right)' = \frac{\partial}{\partial x^\nu}\Lambda^\nu{}_\mu(x). \tag{2.63}$$

The definition of dual basis (2.62) gives us that

$$\Lambda^\mu{}_\rho(x)\Lambda^\rho{}_\nu(x)\big|_{x\to x_0} = \delta^\mu_\nu. \tag{2.64}$$

**Principal Bundle Connection**

For a section $\pi^{-1}$ of the cotangent principal bundle fiber $E(U_{x0})$ on manifold M, the linear connection operator $\nabla_\rho$ is

$$\nabla_\rho dx^\mu = \left.\frac{dx^\mu - \Lambda^\mu{}_\nu(x_0)dx_0^\nu}{x^\rho - x_0^\rho}\right|_{x\to x_0} = \frac{(\delta^\mu_\nu - \Lambda^\mu{}_\nu(x))dx^\nu}{dx^\rho} = -\Gamma^\mu{}_{\nu\rho}(x)dx^\nu, \tag{2.65}$$

then the linear connection operator $\nabla_\rho$ is a functor connects fiber $E(U_x)$ to $E(U_{x0})$

$$\nabla_\rho : E(U_x) \to E(U_{x0}),\, x \to x_0. \tag{2.66}$$

We write connection 1-form as follow

$$\Gamma^\mu{}_\nu(x) = \Gamma^\mu{}_{\nu\rho}(x)dx^\rho, \text{ and} \tag{2.67}$$

the linear connection 1-form operator

$$\nabla = \nabla_\rho dx^\rho,\ \nabla dx^\mu = -\Gamma^\mu{}_\nu(x)dx^\nu. \tag{2.68}$$

The section of the fiber $E^*(U_{x_0})$ of tangent bundle has the connection

$$\nabla_\rho\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\nu}\tilde{\Gamma}^\nu{}_{\mu\rho}(x). \tag{2.69}$$

We omit the $x$ index some places below. The dual relation (2.62) of bases gives us that

$$\nabla_\rho\langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = 0, \quad \Rightarrow \quad \tilde{\Gamma}^\mu{}_{\nu\rho}(x) = -\Gamma^\mu{}_{\nu\rho}(x), \tag{2.70}$$

then

$$\nabla_\rho\left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\nu{}_{\mu\rho}(x)\frac{\partial}{\partial x^\nu}. \tag{2.71}$$

We assumpt that the linear connection operator $\nabla_\rho$ can be defined globally on the manifold M.Under the coordinate transformation in the neighborhood Ux, the transformation rule of the principal bundle connection is derived

$$\begin{aligned}
\nabla_\rho dx'^\mu &= \nabla_\rho \left(\Lambda^\mu{}_\nu(x) dx^\nu\right), \\
\Rightarrow -\Gamma'^\mu{}_{\nu\rho}(x') dx'^\nu &= \left(\frac{\partial \Lambda^\mu{}_\sigma(x)}{\partial x^\rho} - \Lambda^\mu{}_\nu(x)\Gamma^\nu{}_{\sigma\rho}(x)\right) dx^\sigma, \\
\Rightarrow \Gamma'^\mu{}_{\nu\rho}(x')\Lambda^\nu{}_\sigma(x) &= \left(\Lambda^\mu{}_\nu(x)\Gamma^\nu{}_{\sigma\rho}(x) - \frac{\partial \Lambda^\mu{}_\sigma(x)}{\partial x^\rho}\right), \\
\Rightarrow \Gamma'^\mu{}_{\nu\rho}(x') &= \left(\Lambda^\mu{}_\nu(x)\Gamma^\nu{}_{\sigma\rho}(x) - \frac{\partial \Lambda^\mu{}_\sigma(x)}{\partial x^\rho}\right)\Lambda^\sigma{}_\nu(x), \qquad (2.72) \\
\Rightarrow \Gamma'^\mu{}_{\nu}(x') &= \left(\Lambda^\mu{}_\nu(x)\Gamma^\nu{}_{\sigma}(x) - d\Lambda^\mu{}_\sigma(x)\right)\Lambda^\sigma{}_\nu(x), \qquad (2.73)
\end{aligned}$$

such that the cotangent principal bundle $E$(M) has structure of connection preserving left action $G = GL(1 + n, \mathrm{R})$ torsors

$$E(\mathbf{M}) = \frac{G \times E(\mathbf{M})}{G}. \qquad (2.74)$$

**Tangent and cotangent associated bundle**

The tangent associated bundle $TE^*$(M) on manifold M is glued with tangent space on neighborhood of $x$

$$p^* : TE^* \to \mathbf{M}, \qquad TE^*(\mathbf{M}) = \cup TE^*(U_x), \qquad (2.75)$$

the section $p^{*-1}$ of the bundle is a vector field of manifold M

$$V(x) = V^\mu(x)\frac{\partial}{\partial x^\mu}, \qquad \left(V^0(x), V^1(x), \cdots, V^n(x)\right)\big|_{x\to x_0} \in \mathbb{R}^{1+n}. \qquad (2.76)$$

Then the fiber $TE^*(U_{x0})$ of the bundle $TE^*$(M) is isomorphic to $\mathrm{R}^{1+n}$. For definite section of the tangent bundle, there is $GL(1 + n, \mathrm{R})$ freedom to choose the bases of vector in the neighborhood of $x_0$

$$V(x)\big|_{x\to x_0} = V^{\mu\prime}(x)\frac{\partial}{\partial x^\nu}\Lambda^\nu{}_\mu(x)\bigg|_{x\to x_0} = V^\nu(x)\frac{\partial}{\partial x^\nu}\bigg|_{x\to x_0}, \qquad \Lambda^\nu{}_\mu(x)\big|_{x\to x_0} \in GL(1+n, \mathbb{R}). \qquad (2.77)$$

With the help of (2.72) and (2.77), the tangent associated bundle $TE^*$(M) has the struc-ture of connection preserving right action $G = GL(1 + n, \mathrm{R})$ torsors

$$TE^*(\mathbf{M}) = \frac{TE^*(\mathbf{M}) \times G}{G}, \qquad (2.78)$$

the right action structure group $G$ of tangent associated bundle is free and transitive. The contravariant functor $\hat{p}^{*-1}$ of tangent associated bundle $TE^*$(M) is a sheaf on manifold M

$$\hat{p}^{*-1} : \mathbf{M} \to TE^*, \qquad (2.79)$$

where the sheaf $\hat{p}^{*-1}$ are collections of all tangent vector fields on manifold M. The sheaf $\hat{p}^{*-1}$ has structure of connection preserving right action $G = GL(1 + n, \mathrm{R})$ torsors

$$\hat{p}^{*-1} = \frac{\hat{p}^{*-1} \times G}{G}. \tag{2.80}$$

Similarly, the cotangent associated bundle is

$$p : TE \to \mathbf{M}, \qquad TE(\mathbf{M}) = \cup TE(U_x), \tag{2.81}$$

and has the structure of connection preserving left action $G = GL(1+n, \mathrm{R})$ torsors

$$TE(\mathbf{M}) = \frac{G \times TE(\mathbf{M})}{G}, \tag{2.82}$$

the section $p^{-1}$ of the cotangent associated bundle is cotangent vector field (1-form) on manifold M

$$\alpha(x) = \alpha_\mu(x) dx^\mu, \qquad (\alpha_0(x), \alpha_1(x), \cdots, \alpha_n(x))|_{x \to x_0} \in \mathbb{R}^{1+n}. \tag{2.83}$$

The contravariant functor $\hat{p}^{-1}$ is the sheaf of all cotangent vector field on manifold M

$$\hat{p}^{-1} : \mathbf{M} \to TE. \tag{2.84}$$

The sheaf $\hat{p}^{-1}$ have structure of connection preserving left action $G = GL(1+n, \mathrm{R})$ torsors

$$\hat{p}^{-1} = \frac{G \times \hat{p}^{-1}}{}. \tag{2.85}$$

**Lorentz manifold, Riemann geometry and Cartan geometry Metric**

Pseudo Riemann geometry

$$\mathbf{pR} = (\mathbf{M}, g) \tag{3.1}$$

is one of most successful geometry system. The pseudo Riemann geometry pR is a differentiable manifold M with smooth metric tensor $g$

$$g(x) = -g_{\mu\nu}(x) dx^\mu \otimes dx^\nu, \tag{3.2}$$

the metric is symmetric two rank tensor field on manifold M such that the components of metric tensor

$$g_{\mu\nu}(x) = g_{\nu\mu}(x), \tag{3.3}$$

the metric field is non-degenerate, which means, the determinants of metric tensor com-ponents at any point $x_0$ in manifold M are not zero

$$g_v|_{x \to x_0} = \det(g_{\mu\nu}(x))|_{x \to x_0} \neq 0. \tag{3.4}$$

The pseudo Riemann manifold pR has corresponding inverse metric

$$g^{-1}(x) = -g^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}, \tag{3.5}$$

where the dual basis $\frac{\partial}{\partial x^\mu}\big|_{x \to x_0}$ of coordinate $dx^\mu|_{x \to x_0}$ in the neighborhood of $x_0$ satisfy the inner product relation with coordinate

$$\langle \partial_\mu, dx^\nu \rangle|_{x\to x_0} = \delta^\mu_\nu. \tag{3.6}$$

The components of inverse metric $g^{\mu\nu}(x)$ are inverse matrix of metric components $g_{\mu\nu}(x)$ in any point $x_0$

$$g^{\mu\nu}(x) g_{\nu\rho}(x)|_{x\to x_0} = \delta_\rho{}^\mu. \tag{3.7}$$

The metric is compatible with linear connection when

$$\nabla g(x) = 0, \tag{3.8}$$

$$\Rightarrow \frac{\partial g_{\mu\nu}(x)}{\partial x^\rho} - g_{\mu\sigma}(x)\Gamma^\sigma{}_{\nu\rho}(x) - g_{\sigma\nu}(x)\Gamma^\sigma{}_{\mu\rho}(x) = 0. \tag{3.9}$$

We discuss the (1+$n$)-dimensional pseudo Riemann manifold pR with signature $(-, +, +, \cdots)$, Lorentz manifold L, and with signature $(-, -, -, \cdots)$, Riemann manifold R

$$\mathbf{L}, \mathbf{R} \subset \mathbf{pR}. \tag{3.10}$$

Then, $x = (x^\mu) = (x^0, x^q) = (t, \quad x)$, $(q = 1, 2, \cdots, n)$ parameterized the (1+n)-dimensional manifold L and R, and $dx^\mu|_{x\to x0}$ $(\mu = 0, 1, 2, \cdots, n)$ is a coordinate in the neighborhood of $x_0$.

**Curve on Lorentz manifold and Riemann manifold**

The curve $C(\tau)$ on manifold L and R is defined

$$C : \tau \to \mathbf{L}, \mathbf{R}, \qquad \tau \in \mathbb{R}. \tag{3.11}$$

The curve $C(\tau)$ on manifold L and R is an entity then the curve $C(\tau)$ satisfy the repa-rameterization symmetry

$$\begin{array}{ccc} \tau & \xrightarrow{\quad f \quad} & f(\tau) \\ & {\scriptstyle C}\searrow \quad \swarrow{\scriptstyle C'} & \\ & \mathbf{L}, \mathbf{R} & \end{array} \tag{3.12}$$

$$C(\tau) = C'(f(\tau)), \tau, f(\tau) \in \mathbb{R}. \tag{3.13}$$

The metric $g(x)$ on manifold L and R defines a line element of the curve $C(\tau)$

$$ds = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau. \tag{3.14}$$

The length of the any path $C(\tau)$ from $x_0$ point to $x_q$ point on manifold L and R is defined

$$s = \int_{x_0}^{x_q} ds = \int_{x_0}^{x_q} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau. \tag{3.15}$$

The variation of the length $s$ from point $x_0$ to $x_q$ screen out the geodesic curve from point $x_0$ to $x_q$ on manifold L and R

$$\delta s = 0. \tag{3.16}$$

The definition (3.16) of geodesic curve derives that

$$\frac{d^2 x^\mu}{d\tau^2} + \left\{{}^{\mu}_{\nu\rho}\right\}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{3.17}$$

the $\left\{ {}^{\mu}{}_{\nu\rho} \right\}$ is Christoffel symbol and defined by metric components

$$\left\{ {}^{\mu}{}_{\nu\rho} \right\} = \frac{1}{2} g^{\mu\sigma} \left( \frac{\partial g_{\sigma\rho}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \tag{3.18}$$

The $d\tau$ is basis of cotangent vector on curve $C(\tau)$, and the dual basis $\frac{d}{d\tau}$ is defined

$$\left. \langle d\tau, \frac{d}{d\tau} \rangle \right|_{\tau=\tau_0} = 1. \tag{3.19}$$

The restriction of tangent principal bundle $E^*$ from manifold L and R to curve $C(\tau)$ is

$$\begin{array}{ccc} E^*(\mathbf{L}), E^*(\mathbf{R}) & \xrightarrow{restriction} & E^*(U_\tau) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \mathbf{L}, \mathbf{R} & \xrightarrow{restriction} & U_\tau \end{array} \tag{3.20}$$

The objects in $E^*(U_\tau)$ are tangent vector on the curve $C(\tau)$

$$\frac{d}{d\tau} \in E^*(U_\tau), \quad \tau \in \mathbb{R}. \tag{3.21}$$

When the linear connection operator $\nabla_\rho$ acting on tangent vector $\frac{d}{d\tau}$ of curve $C(\tau)$ equals zero, the curve $C(\tau)$ is self-parallel transported

$$\nabla_\rho \left( \frac{d}{d\tau} \right) = 0. \tag{3.22}$$

The definition of self-parallel (3.22) derives that

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}(\tau) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0. \tag{3.23}$$

### Principal bundle on Lorentz manifold and Riemann mani-fold

The freedom to choose $dx^\mu|_{x\to x_0}$ is isomorphic to the fiber $E(U_{x_0})$ of the cotangent prin-cipal bundle E(L) and E(R)→of the (1+n)-dimensional Lorentz manifold L and Riemann manifold R. There is freedom to choose coordinate in the neighborhood of $x_0$

$$E(U_{x_0}) = \{ \Lambda^{\mu}{}_{\nu}(x)|_{x\to x_0} | dx^{\mu\prime}|_{x'\to x_0} = \Lambda^{\mu}{}_{\nu}(x) dx^{\nu}|_{x\to x_0}, \Lambda^{\mu}{}_{\nu}(x)|_{x\to x_0} \in GL(1+n, \mathbb{R}) \}, \tag{3.24}$$

such that the cotangent principal bundle

$$E(\mathbf{L}) = \cup E(U_x), \quad x \in \mathbf{L}, \tag{3.25}$$

and

$$E(\mathbf{R}) = \cup E(U_x), \quad x \in \mathbf{R}, \tag{3.26}$$

has the structure of connection preserving left action $G = GL(1+n, \mathbb{R})$ torsors

$$E(\mathbf{L}) = \frac{G \times E(\mathbf{L})}{G}, \quad E(\mathbf{R}) = \frac{G \times E(\mathbf{R})}{G}. \tag{3.27}$$

For definite metric $g(x)$ of manifold L and R, there is $GL(1 + n,$ R) freedom to choose the coordinate $dx^{\mu}|_{x\to x_0}$ locally to describe the same metric $g(x)$ in the neighborhood of $x_0$

$$g(x)|_{x\to x_0} = -\, g'_{\mu\nu}(x)dx^{\mu\prime}dx^{\nu\prime}\big|_{x\to x_0} = -\, g'_{\mu\nu}(x)\Lambda^{\mu}{}_{\rho}(x)\Lambda^{\nu}{}_{\sigma}(x)dx^{\rho}dx^{\sigma}\big|_{x\to x_0} = -\, g_{\rho\sigma}(x)dx^{\rho}dx^{\sigma}\big|_{x\to x_0}. \tag{3.28}$$

where

$$g'_{\mu\nu}(x)\Lambda^{\mu}{}_{\rho}(x)\Lambda^{\nu}{}_{\sigma}(x)\big|_{x\to x_0} = g_{\rho\sigma}(x)|_{x\to x_0}$$

are used.

For inverse metric, the analyze for tangent principal bundles $E^{*}$(L) and $E^{*}$(R) are similar, and the tangent principal bundle on (1+n)-dimensional manifold L and R has structure of connection preserving right $G = GL(1 + n,$ R) action torsors

$$E^{*}(\mathbf{L}) = \frac{E^{*}(\mathbf{L})\times G}{G}, \quad E^{*}(\mathbf{R}) = \frac{E^{*}(\mathbf{R})\times G}{G}. \tag{3.29}$$

## Orthonomal principal frame bundle and Cartan geometry

The inverse metric $g^{-1}(x)$ in Lorentz manifold L is described by orthonormal frame for-malism ($a, b = 0, 1, 2, \cdots, n$)

$$g^{-1}(x) = -\eta^{ab}\theta_a(x)\theta_b(x), \tag{3.30}$$

where

$$\eta^{ab} = diag(1, -1, -1, \cdots, -1) \tag{3.31}$$

and

$$\theta_a(x) = \theta^{\mu}_{a}(x)\frac{\partial}{\partial x^{\mu}} \tag{3.32}$$

are orthonormal frames and describe gravitational field.The Riemann manifold R be described by inverse metric $g^{-1}(x)$ orthonormal frame formalism as

$$g^{-1}(x) = -I^{ab}\theta_a(x)\theta_b(x), \tag{3.33}$$

where

$$I^{ab} = diag(1, 1, 1, \cdots, 1). \tag{3.34}$$

For definite inverse metric $g^{-1}(x)$, there is $O(1, n)$ freedom to choose the orthonomal frame $\theta^{a}(x)|_{x\to x^0}$ to describe the same metric in the neighborhood of $x_0$

$$\theta^{a\prime}(x)|_{x\to x_0} = \Lambda^{a}{}_{b}(x)\theta^{b}(x)\big|_{x\to x_0}, \qquad \Lambda^{a}{}_{b}(x)|_{x\to x_0} \in O(1, n), \tag{3.35}$$

and

$$g^{-1}(x)\big|_{x\to x_0} = -\,\eta'_{ab}\theta^{a\prime}(x)\theta^{b\prime}(x)\big|_{x\to x_0} = -\,\eta'_{ab}\theta^{c}(x)\Lambda^{a}{}_{c}(x)\theta^{d}(x)\Lambda^{b}{}_{d}(x)\big|_{x\to x_0} = -\,\eta_{ab}\theta^{a}(x)\theta^{b}(x)\big|_{x\to x_0}, \tag{3.36}$$

where

$$\eta'_{ab}\Lambda^a{}_c(x)\Lambda^b{}_d(x)\big|_{x\to x_0} = \eta_{cd}|_{x\to x_0}\,, \qquad \Lambda^a{}_b(x)|_{x\to x_0} \in O(1,n). \tag{3.37}$$

Which means, the fiber $OE^*(U_{x0})$ of orthonormal principal frame bundle $OE^*$(L) is iso-morphic to the orthonormal frame transformation freedom $G = O(1, n)$ (right action) locally

$$OE^*(U_{x_0}) = \left\{ \Lambda^b{}_a(x)\big|_{x\to x_0} \mid \theta'_a(x')|_{x'\to x_0} = \theta_b(x)\Lambda^b{}_a(x)\big|_{x\to x_0}\,, \Lambda^b{}_a(x)\big|_{x\to x_0} \in O(1,n) \right\} \tag{3.38}$$

The orthonomal frame principal bundle is

$$OE^*(\mathbf{L}) = \cup OE^*(U_x), \quad x \in \mathbf{L}, \tag{3.39}$$

The fiber $OE^*(U_{x0})$ of orthonormal principal frame bundle $OE^*$(R) of Riemann manifold R is isomorphic to the orthonormal frame transformation freedom $G = O(1 + n)$ (right action) locally

$$OE^*(U_{x_0}) = \left\{ \Lambda^b{}_a(x)\big|_{x\to x_0} \mid \theta'_a(x')|_{x'\to x_0} = \theta_b(x)\Lambda^b{}_a(x)\big|_{x\to x_0}\,, \Lambda^b{}_a(x)\big|_{x\to x_0} \in O(1+n) \right\} \tag{3.40}$$

The orthonomal frame principal bundle is

$$OE^*(\mathbf{R}) = \cup OE^*(U_x), \quad x \in \mathbf{R}. \tag{3.41}$$

The metric $g(x)$ and $\bar{g}(x)$ can be described by cotangent orthonomal frame (orthono-mal co-frame) formalism as follow

$$g(x) = -\eta_{ab}\theta^a(x)\theta^b(x), \quad \bar{g}(x) = -I_{ab}\theta^a(x)\theta^b(x), \tag{3.42}$$

where

$$\eta_{ab} = diag(1,-1,-1,\cdots,-1), \quad I_{ab} = diag(1,1,1,\cdots,1) \tag{3.43}$$

and

$$\theta^a(x) = \theta^a_\mu(x)dx^\mu \tag{3.44}$$

are cotangent orthonomal frame. It is derived from (3.6) and (3.7) that the cotangent orthonomal frame is dual with tangent orthonomal frame

$$\langle\theta^a(x), \theta_b(x)\rangle|_{x\to x_0} = \delta^a_b \tag{3.45}$$

and

$$\theta^a_\mu(x)\theta^\mu_b(x)\big|_{x\to x_0} = \delta^a_b, \qquad \theta^a_\mu(x)\theta^\nu_a(x)\big|_{x\to x_0} = \delta^\nu_\mu. \tag{3.46}$$

From equation (3.45) we have

$$\Lambda^a{}_c(x)\Lambda^c{}_b(x)|_{x\to x_0} = \delta^a_b. \tag{3.47}$$

The structure group of orthonomal co-frame bundles $OE$(L) and $OE$(R) are $O(1, n)$ and $O(1 + n)$, also.

The orthonomal frame connection coefficients is defined

$$\nabla_\rho\theta_a(x) = \nabla_\rho\left(\theta^\mu_a(x)\frac{\partial}{\partial x^\mu}\right), \tag{3.48}$$

$$\Rightarrow \Gamma^{b}{}_{a\rho}(x)\theta^{\mu}_{b}(x) = \frac{\partial\theta^{\mu}_{a}(x)}{\partial x^{\rho}} + \theta^{\sigma}_{a}(x)\Gamma^{\mu}{}_{\sigma\rho}(x),$$

$$\Rightarrow \Gamma^{b}{}_{a}(x)\theta^{\mu}_{b}(x) = d\theta^{\mu}_{a}(x) + \theta^{\sigma}_{a}(x)\Gamma^{\mu}{}_{\sigma}(x), \tag{3.49}$$

$$\Rightarrow \Gamma^{b}{}_{a\rho}(x) = \left[\frac{\partial\theta^{\mu}_{a}(x)}{\partial x^{\rho}} + \theta^{\sigma}_{a}(x)\Gamma^{\mu}{}_{\sigma\rho}(x)\right]\theta^{b}_{\mu}(x). \tag{3.50}$$

Eliminating the edge term

$$\frac{\partial(\theta^{\mu}_{a}(x)\theta^{b}_{\mu}(x))}{\partial x^{\rho}} = \theta^{\mu}_{a}(x)\frac{\partial\theta^{b}_{\mu}(x)}{\partial x^{\rho}} + \frac{\partial\theta^{\mu}_{a}(x)}{\partial x^{\rho}}\theta^{b}_{\mu}(x), \tag{3.51}$$

can be written as

$$\Gamma^{b}{}_{a\rho}(x) = \left[\theta^{b}_{\sigma}(x)\Gamma^{\sigma}{}_{\mu\rho}(x) - \frac{\partial\theta^{b}_{\mu}(x)}{\partial x^{\rho}}\right]\theta^{\mu}_{a}(x), \tag{3.52}$$

$$\Rightarrow \Gamma^{b}{}_{a}(x) = \left[\theta^{b}_{\sigma}(x)\Gamma^{\sigma}{}_{\mu}(x) - d\theta^{b}_{\mu}(x)\right]\theta^{\mu}_{a}(x). \tag{3.53}$$

The compatible connection condition for orthonomal frame connection coefficients is

$$\eta^{ac}\Gamma^{b}{}_{c\rho}(x) + \eta^{cb}\Gamma^{a}{}_{c\rho}(x) = 0, \tag{3.54}$$

$$\Rightarrow \Gamma^{ba}{}_{\rho}(x) = -\Gamma^{ab}{}_{\rho}(x). \tag{3.55}$$

The connection 1-form on orthonomal frame is defined

$$\Gamma^{a}{}_{c}(x) = \Gamma^{a}{}_{c\rho}(x)dx^{\rho}, \tag{3.56}$$

then we have

$$\nabla\theta_{a}(x) = \Gamma^{b}{}_{a}(x)\theta_{b}(x), \tag{3.57}$$

$$\nabla\theta^{a}(x) = -\Gamma^{a}{}_{b}(x)\theta^{b}(x), \tag{3.58}$$

$$\Gamma^{ab}(x) = -\Gamma^{ba}(x). \tag{3.59}$$

The structure of connection perserving right action $G = O(1, n)$ and $\bar{G} = O(1 + n)$ torsors of orthonomal frame principal bundles

$$OE^{*}(\mathbf{L}) = \frac{OE^{*}(\mathbf{L}) \times G}{G},\ and\ OE^{*}(\mathbf{R}) = \frac{OE^{*}(\mathbf{R}) \times \bar{G}}{\bar{G}}, \tag{3.60}$$

derives that

$$\Gamma'^{b}{}_{a\rho}(x') = \left(\Lambda^{b}{}_{c}(x)\Gamma^{c}{}_{d\rho}(x) - \frac{\partial\Lambda^{b}{}_{d}(x)}{\partial x^{\rho}}\right)\Lambda^{d}{}_{a}(x), \tag{3.61}$$

$$\Rightarrow \Gamma'^{b}{}_{a}(x') = \left(\Lambda^{b}{}_{c}(x)\Gamma^{c}{}_{d}(x) - d\Lambda^{b}{}_{d}(x)\right)\Lambda^{d}{}_{a}(x)\ , \tag{3.62}$$

$$\Rightarrow \Gamma'^{b}{}_{a}(x')\Lambda^{a}{}_{d}(x) = \left(\Lambda^{b}{}_{c}(x)\Gamma^{c}{}_{d}(x) - d\Lambda^{b}{}_{d}(x)\ \right),$$

$$\Rightarrow d\Gamma'^{b}{}_{a}(x')\Lambda^{a}{}_{d}(x) - \Gamma'^{b}{}_{a}(x') \wedge d\Lambda^{a}{}_{d}(x) = d\Lambda^{b}{}_{c}(x) \wedge \Gamma^{c}{}_{d}(x) + \Lambda^{b}{}_{c}(x) \wedge d\Gamma^{c}{}_{d}(x), \tag{3.63}$$

the curvature 2-form is defined

$$\Omega^a{}_b(x) = d\Gamma^a{}_b(x) + \Gamma^a{}_c(x) \wedge \Gamma^c{}_b(x), \tag{3.64}$$

and equation (3.63) derives that the curvature 2-form satisfy the tensor transformation rule

$$\Omega'^a{}_b(x')\Lambda^b{}_d(x) = \Lambda^b{}_c(x)\Omega^c{}_d(x). \tag{3.65}$$

The relation between curvature 2-form $\Omega^a{}_b(x)$ and curvature tensor $R^a{}_{b\mu\nu}(x)$ is

$$\Omega^a{}_b(x) = \frac{1}{2}R^a{}_{b\mu\nu}(x)dx^\mu \wedge dx^\nu, \tag{3.66}$$

where

$$R^a{}_{b\mu\nu}(x) = \frac{\partial\Gamma^a{}_{b\nu}(x)}{\partial x^\mu} - \frac{\partial\Gamma^a{}_{b\mu}(x)}{\partial x^\nu} + \Gamma^a{}_{c\mu}(x)\Gamma^c{}_{b\nu}(x) - \Gamma^a{}_{c\nu}(x)\Gamma^c{}_{b\mu}(x). \tag{3.67}$$

Equation (3.44) bring us that

$$dx^\mu = \theta^\mu_a(x)\theta^a(x), \tag{3.68}$$

after the exterior derivative *d* being acted on equation, the Cartan sturcture equa-tion is derived

$$\begin{aligned} 0 &= d\left(\theta^\mu_a(x)\right) \wedge \theta^a(x) + \theta^\mu_a(x)d\left(\theta^a(x)\right), \\ \Rightarrow d\theta^c(x) &= -\Gamma^c{}_b(x) \wedge \theta^b(x) + \Gamma^c{}_{\mu\nu}(x)dx^\nu \wedge dx^\mu. \end{aligned} \tag{3.69}$$

The trosion 2-form is defined

$$T^c(x) = \frac{1}{2}T^c{}_{\mu\nu}(x)dx^\mu \wedge dx^\nu = -\Gamma^c{}_{\mu\nu}(x)dx^\nu \wedge dx^\mu, \tag{3.70}$$

then the components of torsion is

$$T^c{}_{\mu\nu}(x) = 2\Gamma^c{}_{[\mu\nu]}(x) = \Gamma^c{}_{\mu\nu}(x) - \Gamma^c{}_{\nu\mu}(x), \tag{3.71}$$

and the Cartan sturcture equation is rewritten as

$$d\theta^c(x) + \Gamma^c{}_b(x) \wedge \theta^b(x) + T^c(x) = 0. \tag{3.72}$$

It is easy to prove the torsion satisfy the tensor transformation rule. The exterior deriva-tive *d* acting on equation (3.72) gives us Ricci identity

$$d\Gamma^c{}_b(x) \wedge \theta^b(x) - \Gamma^c{}_b(x) \wedge d\theta^b(x) + dT^c(x) = 0, \tag{3.73}$$

$$\Rightarrow \Omega^c{}_b(x) \wedge \theta^b(x) + \Gamma^c{}_b(x) \wedge T^b(x) + dT^c(x) = 0. \tag{3.74}$$

The equation is Ricci identity in Cartan geometry with torsion, and the components formulation is

$$R^a{}_{[\rho\mu\nu]}(x) + \Gamma^a{}_{\sigma[\rho}(x)T^\sigma{}_{\mu\nu]}(x) + \partial_{[\rho}T^a{}_{\mu\nu]}(x) = 0, \tag{3.75}$$

where

$$\partial_\rho = \frac{\partial}{\partial x^\rho}. \tag{3.76}$$

The exterior derivative *d* acting on Ricci identity (3.74) derives that the Bianchi identity

$$d\Omega^c{}_d(x) - \Omega^c{}_b(x) \wedge \Gamma^b{}_d(x) + \Gamma^c{}_d(x) \wedge \Omega^d{}_b(x) = 0, \tag{3.77}$$

the components formulation of the Bianchi identity is

$$\partial_{[\mu}R^{c}{}_{|d|\nu\rho]}(x) - R^{c}{}_{b[\mu\nu}(x)\Gamma^{b}{}_{|d|\rho]}(x) + \Gamma^{c}{}_{d[\mu}(x)R^{d}{}_{|b|\nu\rho]}(x) = 0. \tag{3.78}$$

The determinants of metric components in the neighborhood of $x_0$ gives us the coor-dinate free volume element $\theta_v(x)$

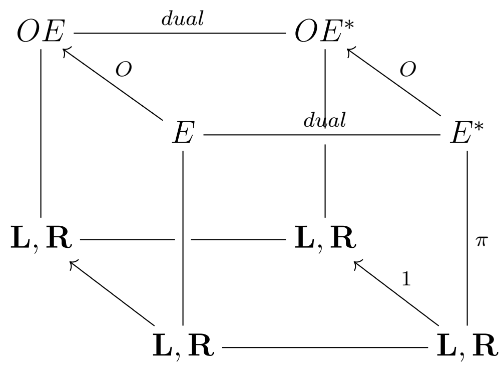
$$g_v(x)|_{x\to x_0} = det\left[g_{\mu\nu}(x)\right]|_{x\to x_0} = det\left[\eta_{ab}\theta^a_\mu(x)\theta^b_\mu(x)\right] = -\,det^2\left[\theta^a_\mu(x)\right]\Big|_{x\to x_0},$$
$$\Rightarrow \theta_v(x)|_{x\to x_0} = det\left[\theta^a_\mu(x)\right]\Big|_{x\to x_0} = \sqrt{-g_v(x)}\Big|_{x\to x_0} \tag{3.79}$$

**Category viewing of principal bundle on Lorentz manifold and Riemann manifold**

The cotangent principal bundle $E^*$(L) and $E^*$(R) are dual to tangent bundle $E$(L) and $E$(R), the orthonomal functor $O$ acting on associatd bundle gives us orthonomal frame bundle and co-frame bundle

$$O : E \to OE, \qquad O : E^* \to OE^*, \tag{3.80}$$

and the commutative diagram of these four kinds of associated bundle on Lorentz manifold L and Riemann manifold R is as follow.




(3.81)

## Clifford algebra and Dirac matrices

*$Cl_{1,n}$* Clifford algebra and Dirac matrices

The $Cl_{1,n}$(R) Clifford algebra has 1 + $n$ generators $\gamma^a$($a$ = 0, 1, 2, · · · , $n$). The Clifford algebra is spanned by the bases as follows

$$Cl_{1,n}(\mathbb{R}) = \text{span} \begin{cases} C^0_{1+n}\ 0-\text{vector} : I, \\ C^1_{1+n}\ 1-\text{vector} : \gamma^{a_1}, \\ C^2_{1+n}\ 2-\text{vector} : \gamma^{a_1}\gamma^{a_2}, & (a_1 < a_2 < a_3 < \cdots < a_{1+n}). \\ C^3_{1+n}\ 3-\text{vector} : \gamma^{a_1}\gamma^{a_2}\gamma^{a_3}, \\ \quad\vdots \\ C^{1+n}_{1+n}\ (1+n)-\text{vector} : \gamma^{a_1}\gamma^{a_2}\gamma^{a_3}\cdots\gamma^{a_{1+n}}, \end{cases} \tag{4.1}$$

The Clifford algebra $Cl_{1,n}$(R) is $2^{1+n}$-dimensional linear space and

$$Cl_{1,n}(\mathbb{R}) = \{\alpha I + \alpha_{a_1}\gamma^{a_1} + \alpha_{a_1a_2}\gamma^{a_1}\gamma^{a_2} + \cdots + \alpha_{a_1a_2\cdots a_{1+n}}\ \gamma^{a_1}\gamma^{a_2}\cdots\gamma^{a_{1+n}}\}, \tag{4.2}$$

where the coefficients before the bases are real valued

$$\alpha, \alpha_{a_1}, \alpha_{a_1a_2}, \cdots, \alpha_{a_1a_2\cdots a_{1+n}} \in \mathbb{R}. \tag{4.3}$$

The matrix representation of generators of Clifford algebra satisfy the restriction

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I_k, \tag{4.4}$$

where $I_k$ is $k \times k$ identity matrix. In physics, the Hermiticity conditions for generators of Clifford algbra can be chosen alwaysly

$$\gamma^a \gamma^{b\dagger} + \gamma^{b\dagger} \gamma^a = 2I^{ab} I_k, \tag{4.5}$$

where $I^{ab}$ is $(1+n) \times (1+n)$ identity matrix. The minimal faithfull matrix representation for $Cl_{1,n}(\mathrm{R})$ gives us the relation

$$2^{1+n} = k \times k \Rightarrow k = \sqrt{2^{1+n}}, \tag{4.6}$$

which means, for any matrix representation of generators of Clifford algebra, there is freedom of $U(k)$ to rotate the matrix representation

$$\gamma^{a\prime} = \psi^\dagger \gamma^a \psi, \quad \psi \in U(k), \tag{4.7}$$

such that the $\gamma^{a'}$ still the generators of Clifford algebra $Cl_{1,n}(\mathrm{R})$. The Dirac matrices can be represented by components formula

$$\gamma^a = \gamma^a{}_{ij} e_j{}^\dagger \otimes e_i = \psi^\dagger \gamma^a \psi \; e_j^\dagger \otimes_j e_i, \; \psi \in U(k), \tag{4.8}$$

$i$

where $e_i (i = 1, 2, \cdots, k)$ are the orthogonal bases expanding (1+n)-dimension complex space $\mathrm{C}^k$ and

$$\mathbf{tr}(e_j^\dagger \otimes e_i) = e_i e_j^\dagger = \delta_{ij}. \tag{4.9}$$

One simple choice of $e_i$ is

$$e_1 = (e^{i\theta_1}, 0, 0, \cdots, 0), \qquad e_2 = (0, e^{i\theta_2}, 0, \cdots, 0), \tag{4.10}$$

$$\cdots \qquad , \qquad e_k = (0, 0, 0, \cdots, e^{i\theta_k}). \tag{4.11}$$

### $Cl_{1,3}(\mathrm{R})$ Clifford algebra and Dirac matrices

Particularly, the solution with

$$n = 3 \; and \; 1, \quad k = 4 \; and \; 2, \tag{4.12}$$

are particular important from the reasons of physics. The corresponding Clifford algebra are $Cl_{1,3}(\mathrm{R})$ and $Cl_{1,1}(\mathrm{R})$. The generators of Clifford algebra $Cl_{1,3}(\mathrm{R})$ is well know Dirac matrices and the bases

$$Cl_{1,3}(\mathbb{R}) = \mathrm{span} \begin{cases} 1 \text{ scalar} : I, \\ 4 \text{ vector} : \gamma^a, \\ 6 \text{ bivector} : \gamma^a\gamma^b, & (a < b < c < d). \\ 4 \text{ pseudovectors} : \gamma^a\gamma^b\gamma^c, \\ 1 \text{ pseudoscalar} : \gamma^a\gamma^b\gamma^c\gamma^d, \end{cases} \tag{4.13}$$

The Weyl representation ($q = 1, 2, 3$) of Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 0 & I_{2\times2}, \\ I_{2\times2} & 0 \end{pmatrix}, \qquad \gamma^q = \begin{pmatrix} 0 & \sigma_q, \\ -\sigma_q & 0 \end{pmatrix}, \tag{4.14}$$

where $\sigma_q$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.15}$$

The components formulation of $\gamma^{a'}$ is

$$\gamma^{a\prime} = \psi^{\dagger}_{li}\gamma^{a}{}_{ij}\psi_{jk}e_k \otimes e^{\dagger}_{l} = \psi^{\dagger}_{i}\gamma^{a}\psi_j e_j \otimes e^{\dagger}_{i}, \quad \psi \in U(4), \tag{4.16}$$

with $i, j = 1, 2, 3, 4$, where $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ are four kinds of Dirac spinors. The element of $U(4)$ group can be presented

$$\psi = e^{-iV_\alpha \mathcal{T}^\alpha}, V_\alpha \in \mathbb{R}, \alpha = 0, 1, 2, \cdots, 15. \tag{4.17}$$

The $T^{\alpha}$ is generators of $U(4)$ group and

$$\mathcal{T}^\alpha = \mathcal{T}^{\alpha\dagger}. \tag{4.18}$$

$Cl_{1,1}$(R) **Clifford algebra**

The generators of Clifford algebra $Cl_{1,1}$(R) can be represented by Pauli matrices, as an example

$$Cl_{1,1}(\mathbb{R}) = \text{span}\begin{cases} 1 \text{ scalar} : I, \\ 2 \text{ vector} : \gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \\ 1 \text{ bivector} : -\sigma_3, \end{cases} \tag{4.19}$$

**Isomorphism between bases of $Cl_{1,3}$(R) and the generators of *U(4)* group**

An isomorphism between the bases of $Cl_{1,3}$(R) and the generators of $U(4)$ group can be constructed as follow. The modified Dirac matrices could be

$$\mathcal{T}^{1,0} = \tilde{\gamma}^0 = \gamma^0, \qquad \mathcal{T}^{1,q} = \tilde{\gamma}^q = i\gamma^q.$$

For modified Dirac matrices

$$\begin{aligned} &\tilde{\gamma}^a\gamma^b + \tilde{\gamma}^b\gamma^c = I^{ab}I_4, \\ &\tilde{\gamma}^{a\dagger} = \gamma^a, \end{aligned}$$

where $I^{ab} = diag(1, 1, 1, 1)$. Then, the isomorphism between the bases of $Cl_{1,3}$(R) and the generators of $U(4)$ group is

$$\begin{aligned} &\mathcal{T}^{1,a} = \tilde{\gamma}^a, \\ &\mathcal{T}^{2,ab} = i\tilde{\gamma}^a\tilde{\gamma}^b, \\ &\mathcal{T}^{3,abc} = i\tilde{\gamma}^a\tilde{\gamma}^b\tilde{\gamma}^c, \qquad (a < b < c < d). \\ &\mathcal{T}^{4,abcd} = \tilde{\gamma}^a\tilde{\gamma}^b\tilde{\gamma}^c\tilde{\gamma}^d, \\ &\mathcal{T}^0 = I_4, \end{aligned}$$

It is easy to see that the constructed $T^{\alpha}$ satisfy

$$\mathcal{T}^{\alpha\dagger} = \mathcal{T}^{\alpha}, \quad Tr\left(\mathcal{T}^{\alpha}\mathcal{T}^{\beta}\right) = \delta^{\alpha\beta}. \tag{4.2}$$

The commutative and anti-commutative relations of constructed $T$ are

$$\begin{aligned}
&[\mathcal{T}^{1,a}, \mathcal{T}^{1,b}] = -2i\mathcal{T}^{2,ab}, && \{\mathcal{T}^{1,a}, \mathcal{T}^{1,b}\} = 0,\\
&[\mathcal{T}^{1,a}, \mathcal{T}^{2,bc}] = 0, && \{\mathcal{T}^{1,a}, \mathcal{T}^{2,bc}\} = 2\mathcal{T}^{3,abc},\\
&[\mathcal{T}^{1,a}, \mathcal{T}^{2,ab}] = 2i\mathcal{T}^{1,b}, && \{\mathcal{T}^{1,a}, \mathcal{T}^{2,ab}\} = 0,\\
&[\mathcal{T}^{1,a}, \mathcal{T}^{3,bcd}] = 2i\mathcal{T}^{4,abcd}, && \{\mathcal{T}^{1,a}, \mathcal{T}^{3,bcd}\} = 0,\\
&[\mathcal{T}^{1,a}, \mathcal{T}^{3,abc}] = 0, && \{\mathcal{T}^{1,a}, \mathcal{T}^{3,abc}\} = 2\mathcal{T}^{2,bc},\\
&[\mathcal{T}^{1,a}, \mathcal{T}^{4,abcd}] = -2i\mathcal{T}^{3,bcd}, && \{\mathcal{T}^{1,a}, \mathcal{T}^{4,abcd}\} = 0,\\
&[\mathcal{T}^{2,ab}, \mathcal{T}^{2,cd}] = 0, && \{\mathcal{T}^{2,ab}, \mathcal{T}^{2,cd}\} = -2\mathcal{T}^{4,abcd},\\
&[\mathcal{T}^{2,ab}, \mathcal{T}^{2,bc}] = 2i\mathcal{T}^{2,ac}, && \{\mathcal{T}^{2,ab}, \mathcal{T}^{2,bc}\} = 0,\\
&[\mathcal{T}^{2,ab}, \mathcal{T}^{3,bcd}] = 2i\mathcal{T}^{3,acd}, && \{\mathcal{T}^{2,ab}, \mathcal{T}^{3,bcd}\} = 0,\\
&[\mathcal{T}^{2,ab}, \mathcal{T}^{3,abc}] = 0, && \{\mathcal{T}^{2,ab}, \mathcal{T}^{3,abc}\} = 2\mathcal{T}^{1,c},\\
&[\mathcal{T}^{2,ab}, \mathcal{T}^{4,abcd}] = 0, && \{\mathcal{T}^{2,ab}, \mathcal{T}^{4,abcd}\} = -2\mathcal{T}^{2,cd},\\
&[\mathcal{T}^{3,abc}, \mathcal{T}^{4,abcd}] = -2i\mathcal{T}^{1,d}, && \{\mathcal{T}^{3,abc}, \mathcal{T}^{4,abcd}\} = 0.
\end{aligned}$$

Explicitly, the constructed generators of $U(4)$ are represented by Dirac matrices

$$\begin{aligned}
&\mathcal{T}^{1} = \gamma^{0}, && \mathcal{T}^{2} = i\gamma^{1},\\
&\mathcal{T}^{3} = i\gamma^{2}, && \mathcal{T}^{4} = i\gamma^{3},\\
&\mathcal{T}^{5} = -\gamma^{0}\gamma^{1}, && \mathcal{T}^{6} = -\gamma^{0}\gamma^{2},\\
&\mathcal{T}^{7} = -\gamma^{0}\gamma^{3}, && \mathcal{T}^{8} = -i\gamma^{1}\gamma^{2},\\
&\mathcal{T}^{9} = -i\gamma^{1}\gamma^{3}, && \mathcal{T}^{10} = -i\gamma^{2}\gamma^{3},\\
&\mathcal{T}^{11} = -i\gamma^{0}\gamma^{1}\gamma^{2}, && \mathcal{T}^{12} = -i\gamma^{0}\gamma^{1}\gamma^{3},\\
&\mathcal{T}^{13} = -i\gamma^{0}\gamma^{2}\gamma^{3}, && \mathcal{T}^{14} = \gamma^{1}\gamma^{2}\gamma^{3},\\
&\mathcal{T}^{15} = -i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}, && \mathcal{T}^{0} = I_{4}.
\end{aligned}$$

and we have

$$[\mathcal{T}^{\alpha}, \mathcal{T}^{\beta}] = f^{\alpha\beta}{}_{\gamma}\mathcal{T}^{\gamma}. \tag{4.21}$$

As an example, the Weyl representation of Dirac matrices could gives us a team of explicit matrix representation of generators of $U(4)$ group.

## Square root Lorentz manifold

### Pair of entities

We define a pair of entities

$$l(x) = i\gamma^{0}(x)\gamma^{a}(x)\theta_{a}(x), \tag{5.1}$$

$$\tilde{l}(x) = i\gamma^{a}(x)\gamma^{0}(x)\theta_{a}(x), \tag{5.2}$$

we call them square root metric of (1 + $n$)-dimensional L and R. This pair of entities describes square root Lorentz manifold rL. Direct calculations show that the definition (5.1) and (5.2) satisfy

$$l^\dagger(x) = -l(x), \quad \tilde{l}^\dagger(x) = -\tilde{l}(x). \tag{5.3}$$

The Dirac matrices on rL has potential to write as follow

$$\gamma^{a\prime}(x) = \gamma^{a\prime}_{ij}(x)e^{\dagger\prime}_j(x) \otimes e'_i(x) = u^\dagger_i(x)\gamma^{a\prime}(x)u_j(x)e^\dagger_j(x) \otimes e_i(x),$$

For any point x0 ∈ L and R

$$\mathbf{tr}(e^\dagger_j(x) \otimes e_i(x))\Big|_{x\to x_0} = e_i(x)e^\dagger_j(x)\Big|_{x\to x_0} = \delta_{ij}. \tag{5.4}$$

One simple choice of ei(x)(i = 1, 2 · · · , k) on manifold is

$$e_1(x) = (e^{i\theta_1(x)}, 0, 0, \cdots, 0), \qquad e_2(x) = (0, e^{i\theta_2(x)}, 0, \cdots, 0), \tag{5.5}$$

$$\cdots \qquad e_k(x) = (0, 0, 0, \cdots, e^{i\theta_k(x)}). \tag{5.6}$$

The bases $e^{\dagger\prime}{}_i(x)\Big|_{x\to x_0}, (i = 1, 2, \cdots, k)$ on rL has U(k) freedom to choose, locally,

$$e_i^{\dagger\prime}(x)\Big|_{x\to x_0} = u_{ij}(x)e_j^\dagger(x)\Big|_{x\to x_0}, u(x)|_{x\to x_0} \in U(k). \tag{5.7}$$

Similarly, there is another local freedom to choose representation of components of Dirac matrices

$$\gamma^{a\prime}(x) = \gamma^{a\prime}_{ij}(x)e^{\dagger\prime}_j(x) \otimes e'_i(x) = u^\dagger_i(x)\gamma^a(x)u_j(x)e^{\dagger\prime}_j(x) \otimes e'_i(x),$$

with

$$\gamma^{a\prime}_{ij}(x)\big|_{x\to x_0} = u^\dagger_{ik}(x)\gamma^a{}_{kl}(x)u_{lj}(x)\Big|_{x\to x_0} = u^\dagger_i(x)\gamma^a(x)u_j(x)\Big|_{x\to x_0}, u(x)|_{x\to x_0} \in U(k'). \tag{5.8}$$

than Then, Lorentz there is extra manifold U(L k′) and× U(k) principal bundle on (1 + n)-dimensional square root Lorentz manifold rL

$$k' = k = \sqrt{2^{1+n}}. \tag{5.9}$$

Under local $U(k')$ × $U(k)$ bases rotation equivalence relation, there still remains $U(k)$ physical freedom

$$\gamma^a(x) = \gamma^a{}_{ij}e^\dagger_j(x) \otimes e_i(x) = \gamma^a{}_{ij}(x)e^\dagger_j \otimes e_i = \psi^\dagger_i(x)\gamma^a\psi_j(x)e^\dagger_j \otimes e_i, \tag{5.10}$$

where

$$\psi(x)|_{x\to x_0} \in U(k) \ (5.11) \tag{5.11}$$

isomorphic to the extra fiber space of associated bundle $UE_1{}^*{}_{,2}$(rL). In the language of mathematic, there are two extra $U(k)$ associated bundles $UE_1{}^*{}_{,2}$(rL) on (1+$n$)-dimensional

square root metric rL than Lorentz manifold L, with structure of left $U(k')$ and right $U(k)$ action torsors

$$UE^*_{1,2}(\mathbf{rL}) = \frac{U(k') \times UE^*_{1,2}(\mathbf{rL}) \times U(k)}{U(k') \times U(k)}, \tag{5.12}$$

where $l(x)$ and $\tilde{l}(x)$ are sections of $UE_1{}^*$(rL) and $UE\ \ _2{}^*$(rL) bundles, respectively. An pair can be written as

$$l(x) = i\gamma^0_{ik}(x)\gamma^a_{kj}(x)e^\dagger_j \otimes e_i\theta_a(x) = i\psi^\dagger_i(x)\gamma^0\gamma^a\psi_j(x)e^\dagger_j \otimes e_i\theta_a(x), \tag{5.13}$$

$$\tilde{l}(x) = i\gamma^a_{ik}(x)\gamma^0_{kj}(x)e^\dagger_j \otimes e_i\theta_a(x) = i\psi^\dagger_i(x)\gamma^a\gamma^0\psi_j(x)e^\dagger_j \otimes e_i\theta_a(x). \tag{5.14}$$

The total structure group of principal bundle $E^*$(rL) on $(1 + n)$-dimensional rL is

$$G = U(k') \times U(k) \times GL(1+n,\mathbb{R}), k = \sqrt{\ 2^{1+n}}. \tag{5.15}$$

The fiber space of associated bundles $UE_1{}^*{}_{,2}$(rL) isomorphic to

$$UE_1{}^*{}_{,2}(U_{x_0}) = U(k) \times GL(1+n,\mathbb{R}), \tag{5.16}$$

and has structure of G-torsors

$$UE^*_{1,2}(\mathbf{rL}) = \frac{U(k') \times UE^*_{1,2}(\mathbf{rL}) \times U(k) \times GL(1+n,\mathbb{R})}{U(k') \times U(k) \times GL(1+n,\mathbb{R})}\ . \tag{5.17}$$

There are two kinds of inverse metric for the pair of entities

$$\bar{g}^{-1}(x) = \frac{1}{4}\mathbf{tr}[l(x)l(x)] = \frac{1}{4}\mathbf{tr}[\tilde{l}(x)\tilde{l}(x)] = -I^{ab}\theta_a(x)\theta_b(x), \tag{5.18}$$

$$g^{-1}(x) = \frac{1}{4}\mathbf{tr}[l(x)\tilde{l}(x)] = \frac{1}{4}\mathbf{tr}[\tilde{l}(x)l(x)] = -\eta^{ab}\theta_a(x)\theta_b(x), \tag{5.19}$$

after using $\gamma^{a\dagger} = \gamma^0\gamma^a\gamma^0$, where $g^{--1}(x)$ and $g^{-1}(x)$ are inverse metric of Riemann manifold R and Lorentz manifold L, respectively. An pair of square root metric for metric of R and L are

$$\bar{l}\qquad(x) = i\gamma_0(x)\gamma_a(x)\theta_\mu(x)dx^\mu, \tag{5.20}$$

$$\bar{\tilde{l}}\qquad(x) = i\gamma_a(x)\gamma_0(x)\theta_\mu{}^a(x)dx^\mu. \tag{5.21}$$

Direct calculation gives us that the definition (5.20) and (5.21) satisfy

$$\bar{l}^\dagger(x) = -\bar{l}(x), \quad \bar{\tilde{l}}^\dagger(x) = -\bar{\tilde{l}}(x). \tag{5.22}$$

The corresponding metric for R and L are

$$\bar{g}(x) = \frac{1}{4}\mathbf{tr}[\bar{l}(x)\bar{l}(x)] = \frac{1}{4}\mathbf{tr}[\bar{\tilde{l}}(x)\bar{\tilde{l}}(x)] = -I_{ab}\theta^a(x)\theta^b(x), \tag{5.23}$$

$$g(x) = \frac{1}{4}\mathbf{tr}[\bar{l}(x)\bar{\tilde{l}}(x)] = \frac{1}{4}\mathbf{tr}[\bar{\tilde{l}}(x)\bar{l}(x)] = -\eta_{ab}\theta^a(x)\theta^b(x). \tag{5.24}$$

The entities pair (5.20) and (5.21) corresponding principal bundle $E$(rL) has total structure group

$$\bar{G} = GL(1+n,\mathbb{R}) \times U(k') \times U(k), \quad k' = k = \sqrt{2^{1+n}}. \tag{5.25}$$

The fiber space of associated bundle UE(rL) isomorphic to

$$UE_{1,2}(U_{x_0}) = U(k) \times GL(1+n,\mathbb{R}), \tag{5.26}$$

and has structure of $\bar{G}$-torsors

$$UE_{1,2}(\mathbf{rL}) = \frac{GL(1+n,\mathbb{R}) \times U(k') \times UE_{1,2}(\mathbf{rL}) \times U(k)}{GL(1+n,\mathbb{R}) \times U(k') \times U(k)}, \tag{5.27}$$

where $\bar{l}(x)$ and $\bar{\tilde{l}}(x)$ are sections of $UE_1$(rL) and $UE_2$(rL) bundles, respectively.

**Connection of extra bundles and gauge field**

The principal bundle connection $W_{\mu ij}(x)$, flavor interaction gauge field, is defined as follow

$$\nabla_\mu e_i^\dagger(x)\Big|_{x\to x_0} = \frac{e_i^\dagger(x) - e_i^\dagger(x_0)}{x^\mu - x_0^\mu}_{x\to x_0} = \frac{(\delta_{ij} - u_{ij}^*(x))e_j^\dagger(x)}{\partial x^\mu}_{x\to x_0} = iW_{\mu ij}(x)e_j^\dagger(x)\Big|_{x\to x_0} \tag{5.28}$$

The conjugate transpose of definition (5.28) gives us that

$$\nabla_\mu e_i(x) = -iW^*{}_{\mu ij}(x)e_j(x), \tag{5.29}$$

The covariant derivative $\nabla_\mu$ acting on (5.4) leads to

$$W_{\mu ij}(x) = W^*{}_{\mu ji}(x). \tag{5.30}$$

The flavor interaction gauge field $W_{\mu ij}(x)$ can be expanded by generators of weak inter-action gauge group U(k)

$$W_{\mu ij}(x) = W_\mu{}^\alpha(x)\mathcal{T}_{ij}{}^\alpha, \alpha = 0, 1, 2, \cdots, k^2 - 1. \tag{5.31}$$

In a word, the flavor interaction gauge field is defined

$$\nabla_\mu e_i(x) = -ie_j(x)W_{\mu ji}(x), \quad \nabla_\mu e_i^\dagger(x) = iW_{\mu ij}(x)e_j^\dagger(x). \tag{5.32}$$

And the gauge fields $W_\mu{}^\alpha(x)$ are real valued

$$W_\mu^\alpha(x) = W_\mu^{\alpha *}(x). \tag{5.33}$$

In Cartan geometry and homology theory, the differential forms are useful. Then, as we use the definition of coordinate free covariant derivative,

$$\nabla = \nabla_\mu dx^\mu, \tag{5.34}$$

it is easy to see

$$\nabla e^\dagger_i(x) = W_{\mu ij}(x)e_i^\dagger(x)dx^\mu = W_{ij}(x)e_i^\dagger(x), \tag{5.35}$$

where

$$W_{ij}(x) = W_{\mu ij}(x)dx^\mu \tag{5.36}$$

is flavor interaction gauge field connection 1-form.
Similarly, the principal bundle connection $V_\mu(x)$, color interaction gauge field, is de-fined as follow

$$\begin{aligned}
\nabla_\mu[\gamma^a(x)]\big|_{x\to x_0} = \frac{\gamma^a(x) - \gamma^a(x_0)}{x^\mu - x_0^\mu}\bigg|_{x\to x_0} &= \frac{\gamma^a(x) - u^\dagger(x)\gamma^a(x)u(x)}{x^\mu - x_0^\mu}\bigg|_{x\to x_0} \\
= \frac{[\gamma^a(x) - u^\dagger(x)\gamma^a(x)] + [u^\dagger(x)\gamma^a(x) - u^\dagger(x)\gamma^a(x)u(x)]}{x^\mu - x_0^\mu}&\bigg|_{x\to x_0} \\
= \frac{[(I_k - u^\dagger(x))\gamma^a(x)] + [u^\dagger(x)\gamma^a(x)(I_k - u(x))]}{x^\mu - x_0^\mu}&\bigg|_{x\to x_0} \\
= i[V_\mu(x)\gamma^a(x) - \gamma^a(x)\bar{V}_\mu(x)]&\big|_{x\to x_0},
\end{aligned} \tag{5.37}$$

we can see

$$\bar{V}_\mu(x) = V_\mu^\dagger(x). \tag{5.38}$$

Then,

$$\nabla_\mu[\gamma^a(x)]\big|_{x\to x_0} = i[V_\mu(x)\gamma^a(x) - \gamma^a(x)V_\mu{}^\dagger(x)]\Big|_{x\to x_0}, \tag{5.39}$$

The conjugate transpose of (5.39) is

$$\nabla_\mu[\gamma^{a\dagger}(x)]\Big|_{x\to x_0} = i[V_\mu(x)\gamma^{a\dagger}(x) - \gamma^{a\dagger}(x)V_\mu{}^\dagger(x)]\Big|_{x\to x_0}, \tag{5.40}$$

As we have Hamiticity condition on square root Lorentz manifold rL

$$\gamma^{a\dagger}(x)\gamma^b(x) + \gamma^{b\dagger}(x)\gamma^a(x)\big|_{x\to x_0} = I^{ab}I_k, \tag{5.41}$$

we act covariant derivative $\nabla_\mu$ on (5.41), after using $\gamma^{a\dagger} = \gamma^0\gamma^a\gamma^0$, it is easy to find out that

$$V_\mu(x) = V_\mu^\dagger(x). \tag{5.42}$$

The $V_\mu$ is $k \times k$ matrix valued field, and can be expanded by generators of $U(k)$ group

$$V_\mu(x) = V_\mu^\alpha(x)\mathcal{T}^\alpha, \quad \alpha = 0, 1, 2, \cdots, k^2 - 1. \tag{5.43}$$

In a word, the color interaction gauge field $V_\mu(x)$ is defined

$$\nabla_\mu(\gamma^a(x)) = i[V_\mu(x)\gamma^a(x) - \gamma^a(x)V_\mu(x)]. \tag{5.44}$$

The conjugate transpose of equation (5.44) is

$$\nabla_\mu(\gamma^{a\dagger}(x)) = i[V_\mu(x)\gamma^{a\dagger}(x) - \gamma^{a\dagger}(x)V_\mu(x)]. \tag{5.45}$$

The connections preserving G and $\bar{G}$-torsors on principal bundles E (rL) and E(rL) lead to the transformation rules of connections $W_{\mu ij}$ (x) and $V_\mu$(x)

$$W'_{\mu ij}(x') = u^*_{ki}(x)W_{\mu kl}(x)u_{lj}(x) + u^*_{ki}(x)\partial_\mu u_{kj}(x), \quad u(x)|_{x\to x_0} \in U(k), \tag{5.46}$$

$$V'_\mu(x') = u(x)V_\mu(x)u^\dagger(x) - (\partial_\mu u(x))u^\dagger(x), \quad u(x)|_{x\to x_0} \in U(k'), \tag{5.47}$$

where

$$u^*_{ji}(x)u_{jk}(x)\big|_{x\to x_0} = \delta_{ik}, \quad u(x)u^\dagger(x)\Big|_{x\to x_0} = I_k. \tag{5.48}$$

The gauge field strength tensors are defined as follows [18]

$$\begin{aligned} F_{\mu\nu ij}(x) &= \partial_\mu W_{\nu ij}(x) - \partial_\nu W_{\mu ij}(x) - iW_{\mu ik}(x)W_{\nu kj}(x) + iW_{\nu ik}(x)W_{\mu kj}(x), \\ H_{\mu\nu}(x) &= \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) - iV_\mu(x)V_\nu(x) + iV_\nu(x)V_\mu(x), \end{aligned}$$

and the transformation rules satisfy

$$F'_{\mu\nu ij}(x') = u^*{}_{ki}(x)F_{\mu\nu kl}(x)u_{lj}(x),\ H'_{\mu\nu}(x') = u(x)H_{\mu\nu}(x)u^\dagger(x). \tag{5.49}$$

From the Hamiticity condition of gauge fields $W_{\mu ij}$ and $V_\mu$, the Hamiticity condition of gauge field strengths are

$$H^\dagger{}_{\mu\nu}(x) = H_{\mu\nu}(x),\ F^*{}_{\mu\nu ij}(x) = F_{\mu\nu ji}(x). \tag{5.50}$$

The gauge field strength tensors can be written by strength 2-form

$$H(x) = \frac{1}{2}H_{\mu\nu}(x)dx^\mu \wedge dx^\nu, \quad F_{ij}(x) = \frac{1}{2}F_{\mu\nu ij}(x)dx^\mu \wedge dx^\nu, \tag{5.51}$$

and

$$F_{ij}(x) = dW_{ij}(x) - iW_{ik}(x) \wedge W_{kj}(x), \tag{5.52}$$

$$H(x) = dV(x) - iV(x) \wedge V(x), \tag{5.53}$$

where

$$V(x) = V_\mu(x)dx^\mu \tag{5.54}$$

is color interaction gauge field connection 1-form. The exterior derivative acting on (5.52) and (5.53) gives us Bianchi identity of strength 2-form

$$dH(x) - iH(x) \wedge V(x) + iV(x) \wedge H(x) \;=\; 0, \tag{5.55}$$

$$dF_{ij}(x) - iF_{ik}(x) \wedge W_{kj}(x) + iW_{ik}(x) \wedge F_{kj}(x) \;=\; 0. \tag{5.56}$$

The tensor formulation of Bianchi identity on this geometry structure as follows

$$\partial_{[\mu}H_{\nu\rho]}(x) \;=\; H_{[\mu\nu}(x)V_{\rho]}(x) - V_{[\mu}(x)H_{\nu\rho]}(x), \tag{5.57}$$

$$\partial_{[\mu}F_{\nu\rho]ij}(x) \;=\; F_{[\mu\nu|ik|}(x)W_{\rho]kj}(x) - W_{[\mu|ik|}(x)F_{\nu\rho]kj}(x). \tag{5.58}$$

**Lagrangian submanifold and Yang-Mills theory in curved space-time**

An pair of equations which satisfying the $U(k') \times U(k)$ gauge invariant, locally Lorentz invariant and generally covariant principles are constructed in (1 + n)-dimensional square root Lorentz manifold rL

$$\mathbf{tr}\nabla[l(x)] = 0, \tag{5.59}$$

$$\mathbf{tr}\nabla[\tilde{l}(x)] = 0, \tag{5.60}$$

those equations are generalized self-parallel transportation principle. Eliminating index $x$, the explicit formulas of equations (5.59) and (5.60) are

$$\left[(i\partial_\mu\bar{\psi}_i - \bar{\psi}_i\tilde{V}_\mu + W_{\mu ij}\bar{\psi}_j)\gamma^a\psi_i + \bar{\psi}_i\gamma^a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji}) + i\bar{\psi}_i\gamma^b\psi_i\Gamma^a{}_{b\mu}\right]\theta^\mu_a = 0,$$

$$\left[(i\partial_\mu\psi^\dagger_i - \psi^\dagger\tilde{V} + W \quad \psi^\dagger)\gamma^a\bar{\psi}^\dagger + \psi^\dagger\gamma^a(i\partial \;\bar{\psi}^\dagger + V\;\bar{\psi}^\dagger - \bar{\psi}^\dagger W \quad) + i\psi^\dagger\gamma^b\bar{\psi}^\dagger\Gamma^a{}_{b\mu}\right]\theta^\mu_a = 0,$$

where

$$\tilde{V}_\mu = \gamma^0 V_\mu\gamma^0, \bar{\psi}^\dagger = \gamma^0\psi. \tag{5.61}$$

The Lagrangians corresponding to equations (5.59) and (5.60) are

$$\mathcal{L} = \bar{\psi}_i\gamma^a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta^\mu_a + \frac{i}{2}\bar{\psi}_i\gamma^b\psi_i\Gamma^a{}_{b\mu}\theta^\mu_a, \tag{5.62}$$

$$\tilde{\mathcal{L}} = \psi^\dagger_i\gamma^a(i\partial_\mu\bar{\psi}^\dagger_i + V_\mu\bar{\psi}^\dagger_i - \bar{\psi}^\dagger_j W_{\mu ji})\theta^\mu_a + \frac{i}{2}\psi^\dagger_i\gamma^b\bar{\psi}^\dagger_i\Gamma^a{}_{b\mu}\theta^\mu_a. \tag{5.63}$$

The last term in Lagrangian (5.62) is Yukawa coupling term $\bar{\psi}_i\phi\psi_i$ and the scalar (Higgs) field is Dirac matrix valued and originated from gravitational field

$$\phi = \frac{i}{2}\gamma^b\Gamma^a{}_{b\mu}\theta^\mu_a. \tag{5.64}$$

Then, the Lagrangian (5.62) describes $U(k') \times U(k)$ Yang-Mills theory in curved space-time. The Lagrangian (5.62) and (5.63) has relation with (5.59) and (5.60)

$$\mathbf{tr}\nabla l(x) = \mathcal{L} - \mathcal{L}^\dagger, \tag{5.65}$$

$$\mathbf{tr}\nabla\tilde{l}(x) = \tilde{\mathcal{L}} - \tilde{\mathcal{L}}^\dagger. \tag{5.66}$$

Then, we say $l(x)$ and $\tilde{l}(x)$ are a Lagrangian submanifolds in $UE_1{}^*$ (rL) and $UE_2{}^*$(rL) , being satisfied, the Lagrangian (5.62) and (5.63) are Hermitian

$$\mathcal{L} = \mathcal{L}^\dagger,$$

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^\dagger.$$

So, the unitary principle (5.67) and (5.68) of quantum field theory consistents with gen-eralized self-parallel transportation principle (5.59) and (5.60). The equations of motion for the Lagrangian (5.62) and (5.63) are

$$\gamma^a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta^\mu_a + \frac{i}{2}\gamma^a\psi_i\Gamma^b{}_{a\mu}\theta^\mu_b = 0, \tag{5.69}$$

$$\gamma^a(i\partial_\mu\bar{\psi}^\dagger_i + V_\mu\bar{\psi}^\dagger_i - \bar{\psi}^\dagger_j W_{\mu ji})\theta^\mu_a + \frac{i}{2}\gamma^a\bar{\psi}^\dagger_i\Gamma^b{}_{a\mu}\theta^\mu_b = 0, \tag{5.70}$$

and these equations conjugate transpose. Then, a pair of Lagrangian (5.62) and (5.63) which describes the U(k′) × U(k) Pati-Salam model type Yang-Mills theory in curved space-time are constructed.

The Yang-Mills Lagrangian for gauge bosons in this geometry can be constructed

$$\mathcal{L}_Y = \frac{-1}{2}\mathbf{tr}\left(H^{\mu\nu}H_{\mu\nu}\right) - \frac{\zeta}{2}F^{\mu\nu}_{ij}F_{\mu\nu ji}, \tag{5.71}$$

where $\zeta \in$ R is constant.

In this geometry framework, the equations can be derived as follows

$$\begin{aligned}\nabla_{[\mu}\nabla_{\nu]}l &= \frac{-1}{2}\left(\bar{\psi}_i\gamma^a\psi_k F_{\mu\nu kj} - F^*_{\mu\nu ki}\bar{\psi}_k\gamma^a\psi_j\right. \\ &+ \left.\bar{\psi}_i\tilde{H}_{\mu\nu}\gamma^a\psi_j - \bar{\psi}_i\gamma^a H_{\mu\nu}\psi_j + \frac{i}{2}\,\bar{\psi}_i\gamma^b\psi_j R^a{}_{b\mu\nu}\right)e^\dagger_j\otimes e_i\theta_a,\end{aligned} \tag{5.72}$$

$$\begin{aligned}\nabla_{[\mu}\nabla_{\nu]}\tilde{l} &= \frac{-1}{2}\left(\bar{\psi}_i\gamma^{a\dagger}\psi_k F_{\mu\nu kj} - F^*_{\mu\nu ki}\bar{\psi}_k\gamma^{a\dagger}\psi_j\right. \\ &+ \left.\bar{\psi}_i H_{\mu\nu}\gamma^{a\dagger}\psi_j - \bar{\psi}_i\gamma^{a\dagger}\tilde{H}_{\mu\nu}\psi_j + \frac{i}{2}\,\bar{\psi}_i\gamma^{b\dagger}\psi_j R^a{}_{b\mu\nu}\right)e^\dagger_j\otimes e_i\theta_a,\end{aligned} \tag{5.73}$$

where $\tilde{H}_{\mu\nu} = \gamma^0 H_{\mu\nu}\gamma^0$. We define $\nabla^2 = \nabla_{[\mu}\nabla_{\nu]}dx^\mu \wedge dx^\nu$ , the equation of motion of this gravity theory is constructed

$$\mathbf{tr}\nabla^2[\tilde{l}(x)l(x)] = 0. \tag{5.74}$$

This equation (5.74) is obviously $U(k') \times U(k)$ gauge invariant, locally Lorentz invariant and generally covariant. The explicit formula of equation (5.74) is

$$R = \frac{i}{4}\left(F_{abij}\psi^\dagger_j(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\psi_i - H_{ab}(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\right), \tag{5.75}$$

where

$$\partial_\mu dx^\nu \otimes dx^\rho\partial_\sigma = \delta^\nu_\mu\delta^\rho_\sigma, \quad dx^\mu \otimes dx^\nu\partial_\rho\partial_\sigma = \delta^\nu_\rho\delta^\mu_\sigma \tag{5.76}$$

are used and

$$F_{abij} = F_{\mu\nu ij}\theta^\mu_a\theta^\nu_b, \quad H_{ab} = H_{\mu\nu}\theta^\mu_a\theta^\nu_b. \tag{5.77}$$

So we define a U(k′) ×U(k) gauge invariant, locally Lorentz invariant, generally covariant Lagrangian

$$\mathcal{L}_g = R\psi^\dagger_i\psi_i - i\left(F_{abij}\psi^\dagger_j(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\psi_i - \psi^\dagger_i H_{ab}(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\psi_i\right). \tag{5.78}$$

$$\mathcal{L}_g = \mathcal{L}^\dagger{}_g. \tag{5.79}$$

The $R\psi_i^\dagger\psi_i$ in Lagrangian (5.78) is the Einstein-Hilbert action. The equation (5.75) and the Einstein tensor can be derived from the Einstein-Hilbert action.

## Conservative currents

The Noether currents for Lagrangian system can be derived from Euler-Lagrangian equa-tions. For action

$$S(\phi_\kappa, \partial_\mu\phi_\kappa) = \int dx^0 \wedge dx^1 \cdots \wedge dx^n\bar{\mathcal{L}}(\phi_\kappa, \partial_\mu\phi_\kappa) = \int dx^0 \wedge dx^1 \cdots \wedge dx^n\theta_v(x)\mathcal{L}(\phi_\kappa, \partial_\mu\phi_\kappa), \tag{5.80}$$

The Euler-Lagrangian equations are

$$\frac{\partial\bar{\mathcal{L}}}{\partial\phi_\kappa} - \partial_\mu\left(\frac{\partial\bar{\mathcal{L}}}{\partial(\partial_\mu\phi_\kappa)}\right) = 0. \tag{5.81}$$

As an example, after careful oberserving of the Lagrangian

$$\bar{\mathcal{L}} = \left[\bar{\psi}_i\gamma^a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta_a^\mu + \frac{i}{2}\bar{\psi}_i\gamma^b\psi_i{\Gamma^a}_{b\mu}\theta_a^\mu\right]\theta_v, \tag{5.82}$$

we set

$$\phi_\kappa = \{\psi_i\}, \tag{5.83}$$

then the Euler-Lagrangian equation gives us four conservative currents equations

$$J_i^\mu \ = \bar{\psi}_i\gamma^a\theta_a^\mu\theta_v, \quad \partial_\mu J_i^\mu = 0. \tag{5.84}$$

Similarly, the conservative currents can be

$$J_i^\mu = \bar{\psi}_i\gamma^a\theta_a^\mu\theta_v, \quad \gamma^a\psi_i\theta_a^\mu\theta_v, \quad \psi_i^\dagger\gamma^a\theta_a^\mu\theta_v, \quad \gamma^a\bar{\psi}_i^\dagger\theta_a^\mu\theta_v, \tag{5.85}$$

for four Lagrangian densities $\mathcal{L}^-$, $\bar{\bar{\mathcal{L}}}^\dagger$, $\bar{\bar{\mathcal{L}}}$ and $\mathcal{L}^\dagger$, respectively. Note that ${\Gamma^a}_{b\mu}{\theta_a}^\mu = \left[\partial_\mu\theta^\mu{}_b + \theta_b^\sigma\Gamma^\mu_{\sigma\mu}\right]$ , we set

$$\phi_\kappa = \{\theta_a^\mu\}, \tag{5.86}$$

then

$$J^b = \bar{\psi}_i\gamma^b\psi_i\theta_v. \tag{5.87}$$

**Sheaf quantization and path integral quantization**

Sheaf quantization

${UE_1}^*(\mathbf{rL})$ and ${UE_2}^*(\mathbf{rL})$ , are two associated bundles on square root Lorentz manifold rL

$$\begin{array}{ccc} UE_1^* & \xrightarrow{\gamma^a\to\gamma^{a\dagger}} & UE_2^* \\ & {\searrow}^{\pi^*} \quad {\swarrow}^{\pi^*} & \\ & \mathbf{rL} & \end{array} \tag{6.1}$$

Pair of entities $l(x)$ and $\tilde{l}(x)$ are sections of the bundles ${UE^*}_1(\mathbf{rL})$ and ${UE_2}^*(\mathbf{rL})$, respectively

$$\begin{array}{ccc} l(x) & \xrightarrow{\gamma^a\to\gamma^{a\dagger}} & \tilde{l}(x) \\ & {\nwarrow}^{\pi^{*-1}} \quad {\nearrow}^{\pi^{*-1}} & \\ & \mathrm{rL} & \end{array} \tag{6.2}$$

In mathematic, the sheaf space SH1(rL) and SH2(rL) are spanned by collection of one kind sections of the bundles

$$\begin{array}{ccc} SH_1 & \xrightarrow{\gamma^a\to\gamma^{a\dagger}} & SH_2 \\ & {\nwarrow}^{\hat{\pi}^{*-1}} \quad {\nearrow}^{\hat{\pi}^{*-1}} & \\ & \mathbf{rL} & \end{array} \tag{6.3}$$

The sheaf spaces $SH1$(rL) and $SH2$(rL) are dual to each other. The superposition prin-ciple in quantum mechanics tells us that if the quantum state $|\Psi\rangle_1$ and $|\Psi\rangle_2$ exist, the superposition state

$$|\Psi\rangle = \alpha_1|\Psi\rangle_1 + \alpha_2|\Psi\rangle_2, \alpha_1, \alpha_2 \in \mathbb{C}, \tag{6.4}$$

exist also. For pure state, the sheaf space $SH1$(rL) and $SH2$(rL) valued entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$ can be defined

$$\hat{l}(x) = \sum_\kappa \alpha_\kappa(x)\alpha_\kappa^*(x)|\kappa, x\rangle\langle\kappa, x|l_\kappa(x), \tag{6.5}$$

$$\hat{\tilde{l}}(x) = \sum_\kappa \alpha_\kappa(x)\alpha_\kappa^*(x)|\kappa, x\rangle\langle\kappa, x|\tilde{l}_\kappa(x), \tag{6.6}$$

with the quantum field theory quantum state

$$|\Psi(x)\rangle = \sum_\kappa \alpha_\kappa(x)|\kappa, x\rangle, \tag{6.7}$$

where κ is sheaf space index and evaluated in an abelian group. The orthgonalization relation of bases in sheaf spaces $SH_1$(rL), $SH_2$(rL) and probability complete formulas can be defined

$$\langle \kappa, x|\kappa', x'\rangle = \delta(x-x')\delta(\kappa-\kappa'), \tag{6.8}$$

$$\int \mathbf{tr}\left(|\Psi(x)\rangle\langle\Psi(x)|\right)dx = \int\sum_{\kappa}\alpha_{\kappa}(x)\alpha^{*}_{\kappa}(x)dx = 1, \tag{6.9}$$

where

$$dx = dx^1 \wedge \cdots \wedge dx^n. \tag{6.10}$$

The exterior derivative acting on (6.9)

$$d\left[\mathbf{tr}\int\left(|\Psi(x)\rangle\langle\Psi(x)|\right)dx\right] = \int \mathbf{tr}\left[d\left(|\Psi(x)\rangle\langle\Psi(x)|\right)\right]dx = 0, \tag{6.11}$$

gives us the Schordinger equations

$$i\frac{\partial|\Psi(x)\rangle}{\partial t} = \hat{H}(x)|\Psi(x)\rangle, \quad \hat{H}(x) = \hat{H}^{\dagger}(x), \tag{6.12}$$

$$i\frac{\partial|\Psi(x)\rangle}{\partial x^q} = \hat{P}_q(x)|\Psi(x)\rangle, \quad \hat{P}_q(x) = \hat{P}^{\dagger}_q(x). \tag{6.13}$$

The quantum state of quantum field theory might be presented by mixed state

$$\rho(x) = \sum_{\kappa}\eta_{\kappa}(x)|\kappa, x\rangle\langle\kappa, x|. \tag{6.14}$$

Further, the correspongding sheaf valued entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$ can be written

$$\hat{l}(x) = \sum_{\kappa}\eta_{\kappa}(x)|\kappa, x\rangle\langle\kappa, x|l_{\kappa}(x), \tag{6.15}$$

$$\hat{\tilde{l}}(x) = \sum_{\kappa}\eta_{\kappa}(x)|\kappa, x\rangle\langle\kappa, x|\tilde{l}_{\kappa}(x), \tag{6.16}$$

where $\eta_{\kappa}(x)$ are probability density of corresponding section $l_{\kappa}(x)$ and $\tilde{l}_{\kappa}(x)$. The proba-bility complete formulas in mixed state case is

$$\int \mathbf{tr}\rho(x)dx = \int\sum_{\kappa}\eta_{\kappa}(x)dx = 1. \tag{6.17}$$

The sheaf spaces $SH1$(rL) and $SH2$(rL) are linear spaces, which means, for example, any two entitis in $SH1$(rL), there is a entity equals to the mixing of the two entitis

$$\begin{aligned}&\hat{l}(x) = \eta_1(x)\hat{l}_1(x) + \eta_2(x)\hat{l}_2(x); \hat{l}_1(x), \hat{l}_2(x) \in SH_1(\mathbf{rL})\\ \Rightarrow\ &\hat{l}(x) \in SH_1(\mathbf{rL}),\end{aligned} \tag{6.18}$$

where

$$\int dx\eta_1(x), \int dx\eta_2(x) \in [0,1], \tag{6.19}$$

and

$$\int dx\left[\eta_1(x) + \eta_2(x)\right] = 1. \tag{6.20}$$

We call it sheaf quantization which switching study objects from single section to one kind of possible sections of the bundle. Sheaf quantization method find out a pair of linear space $SH1$(rL) and $SH2$(rL) even the rL is curved space-time, sheaf quantization method consistent with superposition principle. The equations of motion for entities $\hat{l}(x)$ and $\hat{\tilde{l}}(x)$ after sheaf quantization are

$$\mathbf{tr}\nabla[\hat{l}(x)] = 0, \quad \mathbf{tr}\nabla^2[\hat{\tilde{l}}(x)\hat{l}(x)] = 0. \tag{6.21}$$

The corresponding total Lagrangian density is

$$\hat{\mathcal{L}} = \sum_{\kappa}\eta_{\kappa}(\mathcal{L}_{\kappa} + g\mathcal{L}_{g,\kappa} + \tilde{g}\mathcal{L}_{Y,\kappa}), \tag{6.22}$$

where $g$, $\tilde{g}$ are Lagrange multipliers and

$$g, \tilde{g} \in \mathbb{R}. \tag{6.23}$$

**The relations between sheaf quantization and path integral quantization**

The Schordinger equation (6.12) derives that

$$|\Psi(t+\Delta t, x^q)\rangle = U(t+\Delta t, t)|\Psi(t, x^q)\rangle = e^{-i\hat{H}(t,x^q)\Delta t}|\Psi(t, x^q)\rangle, \tag{6.24}$$

$$\text{where } t = x^0,\ q = 1, 2, \cdots, n \text{ and } \Delta t \to 0. \tag{6.25}$$

By using the orthgonalization relation (6.8) of bases in the sheaf spaces $SH_1(rL)$ and $SH_2(rL)$, and we choose the orthgonal bases to be $|\phi_\kappa(t, x^q)$ to span the quantum state of quantum field theory

$$|\Psi(t, x^q)\rangle = \sum_\kappa \alpha_\kappa(t, x^q)\phi_\kappa(t, x^q)|0\rangle = \sum_\kappa \alpha_\kappa(t, x^q)|\phi_\kappa(t, x^q)\rangle, \tag{6.26}$$

the equation (6.24) can be written

$$\begin{aligned}\alpha_{\kappa''}(t+\Delta t, x^q) &= \sum_\kappa \langle \phi_{\kappa''}(t+\Delta t, x^q)|e^{-i\hat{H}(t,x^q)\Delta t}|\phi_\kappa(t, x^q)\rangle \alpha_\kappa(t, x^q) \\ &= \sum_{\kappa,\kappa'} \int d\pi_{\kappa'}(t+\Delta t, x^q)\langle \phi_{\kappa''}(t+\Delta t, x^q)|\pi_{\kappa'}(t+\Delta t, x^q)\rangle \\ &\quad \langle \pi_{\kappa'}(t+\Delta t, x^q)|e^{-i\hat{H}(t,x^q)\Delta t}|\phi_\kappa(t, x^q)\rangle \alpha_\kappa(t, x^q).\end{aligned} \tag{6.27}$$

We know the relation of canonical position $|\phi_\kappa(t, x^q)\rangle$ and momentum $|\pi_\kappa(t, x^q)\rangle$

$$\langle \phi_\kappa(t, x^q)|\pi_{\kappa'}(t, x^q)\rangle = e^{i\pi_\kappa(t,x^q)\phi_\kappa(t,x^q)}\delta_{\kappa\kappa'}, \tag{6.28}$$

this is second quantized version of

$$\langle x|p\rangle = e^{ipx}, \tag{6.29}$$

then

$$\begin{aligned}\alpha_\kappa(t+\Delta t, x^q) &= \sum_\kappa \int d\pi_\kappa(t+\Delta t, x^q)e^{i\pi_\kappa(t+\Delta t,x^q)[\phi_\kappa(t+\Delta t,x^q)-\phi_\kappa(t,x^q)]}e^{-i\hat{H}(t,x^q)\Delta t}\alpha_\kappa(t, x^q) \\ &= \sum_\kappa \int d\pi_\kappa(t+\Delta t, x^q)e^{i\pi_\kappa(t+\Delta t,x^q)\dot{\phi}_\kappa(t,x^q)\Delta t}e^{-i\hat{H}(t,x^q)\Delta t}\alpha_\kappa(t, x^q)\end{aligned} \tag{6.30}$$

There is Legendre transformation between Hamiltonian and Lagrangian

$$\hat{L} = \sum_\kappa \pi_\kappa \dot{\phi}_\kappa - \hat{H} = \int dx\theta_v \hat{\mathcal{L}}, \tag{6.31}$$

then

$$\begin{aligned}\alpha_\kappa(t+\Delta t, x^q) &= \sum_\kappa \int d\pi_\kappa(t+\Delta t, x^q)e^{i(\pi_\kappa\dot{\phi}_\kappa - H)\Delta t}\alpha_\kappa(t, x^q) \\ &= \sum_\kappa \int_{t'=t+\Delta t} d\pi_\kappa(t', x^q)e^{i\hat{L}\Delta t}\alpha_\kappa(t, x^q) \\ &= \sum_\kappa \int_{t'=t+\Delta t} d\pi_\kappa(t', x^q)e^{i\int \omega\hat{\mathcal{L}}}\alpha_\kappa(t, x^q),\end{aligned} \tag{6.32}$$

where ω is volume form

$$\omega = \theta_v dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n \tag{6.33}$$

then Then, the transition amplitude can be defined through path integral formula

$$\alpha_\kappa(t, x^q) = \sum_\kappa \int_{t' \in (t_0, t)} D\pi_\kappa(t', x^q) e^{i \int \omega \hat{\mathcal{L}}[\phi_\kappa(t', x^q), \partial_\mu \phi_\kappa(t', x^q)]} \alpha_\kappa(t_0, x^q). \quad (6.34)$$

This section of proof shows that, the sheaf quantization method is consistent with path integral method even for quantum field t heory i n c urved s pace-time. A s w e are using the second quantized canonical $|\phi_K(t, x^q)\rangle$ and momentum $|\pi_K(t, x^q)\rangle$, the manifold after sheaf quantization and path integral quantization should be second quantized version symplectic manifold.

## Conclusion and Discussion

The existence of extra bundles on square root Lorentz manifold and the self-parallel transportation principle lead us to the Pati-Salam model in curved space-time and the Einstein-Cartan gravity. The relations between sheaf quantization method and path inte-gral quantization method is proved. The prove shows that the sheaf quantization method is consistent with path integral method even the base manifold with curvature.

The discussions about homology theory, homotopy theory, characteristic class in square root Lorentz manifold will be wonderful. The global solutions of square root Lorentz manifold with topologies $S^1 \times S^3$ and $S^1 \times S^1$ of base manifold are interesting. The micro support language of sheaf of square root Lorentz manifold might trigger a meaningful collision between mathematic theory and physical theory.

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