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OSCILLATORY AND NON OSCILLATORY PROPERTIES OF RICCATI TYPE DIFFERENCE EQUATIONS

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ABSTRACT

In this paper we study the Oscillatory and Non Oscillatory Properties of Difference Equations of Riccati type. The Riccati's difference equation is defined by –

$$\Delta(p(n)\Delta u(n) + r(n)u(n+1) = 0,\ n \in N$$

Key words: Difference equation, Sequence, Oscillation and Non Oscillation, Double summation, Riccati's equation.

INTRODUCTION

In the recent years there has been a lot of interest in the study of oscillatory and non oscillatory properties of difference equations.

We are concern with the oscillatory and Non Oscillatory properties of Riccati's difference equation of the form –

$$\Delta(p(n)\Delta u(n) + r(n)u(n+1) = 0, \quad n \in N \qquad \ldots(1)$$

where the functions p and r are defined on N, and $p(n) > 0$ for all $n \in N$. The difference equation (1) is equivalent to $p(n)u(n+1) + p(n-1)u(n-1) = q(n)u(n)$, $n \in N$ with $q(n) = p(n) + p(n-1) - r(n-1)$. If u(n) is a solution of (1) with $u(n)u(n+1) > 0$ for all $n \in N(a)$, then the Riccati type transformation we let $v(n) = p(n)\Delta u(n)/u(n)$. Since $v(n) + p(n) = p(n)u(n+1)/u(n) > 0$, this leads to Riccati type difference equation.

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$$\Delta v(n)+\frac{v(n)v(n+1)+r(n)v(n)}{p(n)}+r(n)=0,\ n\in N(a) \qquad \ldots(2)$$

which is same as –

$$\Delta v(n)+\frac{v^2(n)}{v(n)+p(n)}+r(n)=0,\ n\in N(a) \qquad \ldots(3)$$

Definition: 1 (Difference equation): An equation which expresses a value of a sequence as a function of the other terms in the sequence is called a difference equation.

Definition: 2 (Order of a Difference Equation): The difference between the largest and smallest arguments appearing in the difference equation is called its order.

Example, $y_{k+1}-3y_k+y_{k-1}=e^{-k}$, (2nd order Difference Equation)

Definition: 3 (Oscillation and Non Oscillation): The sequence y is said to be oscillatory around a $(a\in R)$ if there exists an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $(y_{n_k}-a)\,(y_{n_{k+1}}-a)\le 0$ for all $k\in N$. If it is not Oscillatory then the difference equation is non oscillatory.

RESULTS AND DISCUSSION

Theorem 1: The difference equation (1) is non oscillatory if and only if there exists a function $w(n)$ defined on N with $w(n)>-p(n), k\in N(a)$ for some $a\in N$, satisfying.

$$\Delta w(n)+\frac{w(n)w(n+1)+r(n)w(n)}{p(n)}+r(n)\le 0 \qquad \ldots(4)$$

or equivalently

$$\Delta w(n)+\frac{w^2(n)}{w(n)+p(n)}+r(n0\le 0 \qquad \ldots(5)$$

Proof: Since the necessary part is obvious, we need to prove only the sufficient part. For this, let $z(a)=1,\ z(n)=\prod_{l=a}^{n-1}\left(1+\frac{w(l)}{p(l)}\right),\ n\in N(a+1)$ then $z(n)>0$ for all $n\in N(a)$ and

$$\Delta(p(n)\Delta z(n))+r(n)z(n+1)\le 0 \qquad \ldots(6)$$

Therefore, by definition the difference equation (1) is Non oscillatory if and only if there exists a function $v(n)$ satisfying $v(n) > 0$ and $\Delta(p(n)\Delta v(n)) + r(n)v(k+1) \leq 0$ for all sufficiently large $n \in N$.

Theorem 2

$$\lim_{n\to\infty} \sup n^{\frac{-3}{2}} \sum_{l=0}^{n} p(l) < \infty \qquad \ldots(7)$$

and the difference equation (1) is non oscillatory. Then, the following are equivalent.

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} \sum_{\tau=0}^{l} r(\tau)$ exists ...(8)

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} \sum_{\tau=0}^{l} r(\tau) > -\infty$ ...(9)

(iii) For any non oscillatory solution $u(n)$ of (1) with $u(n)u(n+1) > 0$, $n \in N(a)$, the function $v(n) = p(n)\Delta u(n)/u(n)$, $n \in N(a)$ satisfies.

$$\sum_{l=a}^{\infty} \frac{v^2(l)}{v(l) + p(l)} < \infty \qquad \ldots(10)$$

Proof: Clearly (i) implies (ii). To show that (ii) implies (iii) suppose to the contrary that there is a non oscillatory solution $u(n)$ of (1) such that $v(n) = p(n)\Delta u(n)/u(n) > -p(n)$ for all $n \in N(a)$ and

$$\sum_{l=a}^{\infty} \frac{v^2(l)}{v(l) + p(l)} = \infty \qquad \ldots(11)$$

From (3), we have

$$v(n+1) + \sum_{l=a}^{n} \frac{v^2(l)}{v(l) + p(l)} + \sum_{l=a}^{n} r(l) = v(a) \qquad \ldots(12)$$

and therefore for all $n \in N(a)$

$$\frac{1}{n}\sum_{l=a}^{n}(-v(l+1)) = \frac{1}{n}\sum_{l=a}^{n}\sum_{\tau=a}^{l}\frac{v^2(\tau)}{v(\tau)+p(\tau)} \quad \ldots(13)$$

$$= \frac{1}{n}\sum_{l=a}^{n}\sum_{\tau=a}^{l} r(\tau) - \left(\frac{k-a+1}{n}\right)v(a)$$

From (9), (11) and (13), we obtain $\lim_{n\to\infty}\frac{1}{n}\sum_{l=a}^{n}(--v(l+1)) = \infty$ and hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=a}^{n}\left|v(l)\right| = \infty \quad \ldots(14)$$

Let $p(n) = v^2(n)/v(n) + p(n),\ n \in N(a)$. Then, $p(n) \geq 0$ and $p(n) = 0$, if and only, if $v(n) = 0$. Let $A(n) = v^2(n)/p(n)$, if $v(n) \neq 0$ and $A(k) = 0$, if $v(n) = 0$. Then, we have $p(n) \geq A(n) - v(n)$ and hence

$$n^{\frac{-3}{2}}\sum_{l=a}^{n} p(l) \geq n^{\frac{-3}{2}}\sum_{l=a}^{n} A(l) + n^{\frac{-3}{2}}\sum_{l=a}^{n}(-v(l)) \quad \ldots(15)$$

Thus , in the equation (7) and $A(n) \geq 0$ it follows that –

$$\limsup_{n\to\infty} n^{\frac{-3}{2}}\sum_{l=a}^{n}(-v(l)) < \infty \quad \ldots(16)$$

Therefore, on dividing both sides of (13) by $n^{\frac{1}{2}}$, and in the resulting equation using (9) and (16) leads to –

$$\limsup_{n\to\infty} n^{\frac{-3}{2}}\sum_{l=a}^{n}\sum_{\tau=a}^{l} P(\tau) < \infty \quad \ldots(17)$$

Now since

$$k^{\frac{-1}{2}}\sum_{l=a}^{n} P(l) = n^{\frac{-3}{2}}\, n\sum_{l=a}^{n} P(l) \leq n^{\frac{-3}{2}}\sum_{l=a}^{2n}\sum_{\tau=a}^{l} P(\tau)$$

$$= 2^{\frac{3}{2}}(2n)^{\frac{-3}{2}}\sum_{l=a}^{2n}\sum_{\tau=a}^{l} P(\tau)$$ from (17) we have –

$$\limsup_{n\to\infty} n^{-\frac{1}{2}} \sum_{l=a}^{n} P(l) < \infty \qquad \ldots(18)$$

From (18) there is an M > 0 such that –

$$\left(\sum_{l=a}^{n} |v(l)|\right)^2 = \left(\sum_{l=a}^{n} (A(l)P(l))^{\frac{1}{2}}\right)^2 \leq \sum_{l=a}^{n} A(l) \sum_{l=a}^{n} P(l) \leq Mn^{\frac{1}{2}} \sum_{l=a}^{n} A(l)$$

Therefore, it follows that –

$$n^{\frac{-3}{2}} \sum_{l=a}^{n} A(l) \geq \frac{1}{M}\left(\frac{1}{n}\sum_{l=a}^{n} |v(l)|\right)^2$$

and hence from (14) – (16), we have –

$$\lim_{n\to\infty} n^{\frac{-1}{2}} \sum_{l=a}^{n} p(l) = \infty$$

which contradicts (7).

Finally, we shall show that (iii) implies (i). Let $v(n)$ be as in (iii) and let $B(n) = \sum_{l=a}^{n} |v(l)|$.

Then, we have –

$$\left(\sum_{l=a}^{n} v(l)\right)^2 \leq B^2(l) = \left(\sum_{l=a}^{n} [P(l)(v(l) + p(l))]^{\frac{1}{2}}\right)^2$$

$$\leq \sum_{l=a}^{n} P(l) \sum_{l=a}^{n} (v(l) + p(l)) \leq \; l \left(B(n) + \sum_{l=a}^{n} p(l)\right) \leq 2L \max\left\{B(n), \sum_{l=a}^{n} p(l)\right\}$$

where $L = \sum_{l=a}^{\infty} P(l)$. Hence, we have $B(n) \leq \max\left\{2L, \left(2L\sum_{l=a}^{n} p(l)\right)^{\frac{1}{2}}\right\}$

Thus, from (7) it follows that $\lim_{n\to\infty}\left(\frac{1}{n}\right)B(n)=0$, so that $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{l=a}^{n}\left(-v(l+1)\right)=0.$ The result (i) now follows by letting $n\to\infty$ in (13).

Theorem 3

Assume that
$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}p(l)<\infty \qquad \ldots(19)$$

$$\liminf_{n\to\infty}\sum_{l=0}^{n}\sum_{\tau=}^{l}r(\tau)=-\infty \qquad \ldots(20)$$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}\sum_{\tau=0}^{l}r(\tau)>-\infty \qquad \ldots(21)$$

Then, the difference equation (1) is oscillatory.

Proof: Suppose to contrary that (1) is non oscillatory and let $u(n)$ be any non oscillatory solution. Let $v(n)=p(n)\Delta u(n)/u(n)$ for $n\in N(a)$. Since condition (7) follows from (19), Theorem 2 and (20) imply that (11) holds. But, from (13) we have –

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{l=a}^{n}\left(-v(l+1)\right)\geq\limsup_{n\to\infty}\frac{1}{n}\sum_{l=a}^{n}\sum_{\tau=a}^{l}\frac{v^2(\tau)}{v(\tau)+p(\tau)}+\limsup_{n\to\infty}\frac{1}{n}\sum_{l=a}^{n}\sum_{\tau=a}^{l}r(\tau)-v(a)=\infty,$$

which is impossible from $-v(l+1)<p(l+1)$ and (19).

Theorem 4: If there exist two sequences $\{n_l\}$ and $\{m_l\}$ of integers with $m_l\geq n_l+1$ such that –

$$n_l\to\infty \text{ as } l\to\infty$$

and
$$\sum_{\tau=n_l}^{m_l-1}r(\tau)\geq p(n_l)+p(m_l) \qquad \ldots(22)$$

then (1) is oscillatory.

Proof: Suppose that (1) is non oscillatory. Then, there exists a non oscillatory solution u(n) such that $u(n)u(n+1)>0$ for all $n\in N(a)$ for some $a\in N$. $v(n)=p(n)\Delta u(n)/u(n)$. Then, $v(n)$ satisfies (3) Let $v(n)>-p(n)$ for all k $n\in N(a+1)$ and then this contradiction will prove the theorem.

From (3) $r(a) = v(a) - v(a+1) - \dfrac{v^2(a)}{v(a)+p(a)} < p(a+1) + \dfrac{v(a)p(a)}{v(a)+p(a)}$

$$= p(a+1) + p(a) - \frac{p^2(a)}{v(a)+p(a)} < p(a+1) + p(a)$$

Therefore, (22) holds for $n = a+1$. For any $n \in N(a+2)$ from (3) we have –

$$\sum_{l=a+1}^{n-1} r(l) = v(a+1) - v(n) - \sum_{l=a+1}^{n-1} \frac{v^2(l)}{v(l)+p(l)} < u(a+1) + p(n)$$

However, since $v(a+1) = p(a)\left(1 - \dfrac{u(a)}{u(a+1)})\right) - \tau(a) < p(a) - r(a)$, it follows that immediately.

CONCLUSION

By using theorems (1) and (2) the given Riccati difference equation (1) is Non oscillatory and by theorems (3) and (4) the given equation (1) is Oscillatory.

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