Composite Materials : An International Journal

Full Paper

CMAIJ, 1(1), 2016 [025-033]

On the stability of dufour- driven generalized double-diffusive shear flows

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ABSTRACT

The present paper investigates the stability of Dufour –driven generalized double-diffusive shear flows. The physical configuration is that of a horizontal layer of an incompressible inviscid heat conducting fluid of zero electrical resistivity in which there is a differential streaming U(z) in the horizontal direction and density variation $\rho_0 f(z)$ in the vertical direction while the entire system is confined between two horizontal boundaries of different but uniform temperature and concentration with the temperature and the concentration of the lower boundary greater than that of the upper one or vice-versa, ρ_0 being a positive constant having the dimension of density and U(z) and $f(z)$ being continuous functions of the vertical coordinate z with $\frac{df}{dz} < 0$ everywhere in the flow domain. Sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system for the cases when the temperature and the concentration make opposing contributions to the vertical density gradient.

KEYWORDS

Double-diffusive convection;
Non-homogeneous shear flows;
Dufour –effect.

INTRODUCTION

The stability of parallel shear flow of an inviscid non-homogeneous fluid with stable density stratification to infinitesimal non-divergent disturbances has pervaded the scientific literature in the recent past on account of its importance in the fields of meteorology and oceanography etc. For a broad view of the subject one may refer to the fundamental works of Taylor[1], Goldstein[2], Drazin[3], Miles[4], Howard[5] and others on the stability of non-homogeneous shear flows. In the mathematical model of the problem considered by these authors, the fluid is taken to be initially non-homogeneous without assigning any reason for the cause of this initial non-homogeneity. However, the initial non-homogeneity may be due to variable temperature or concentration or some other cause. Diffusion effects which tend to produce these changes in the density of an individual fluid particle in the course of motion are ignored in these investigations. Therefore, it became important to investigate the problem by retaining the initial non-homogeneity and also taking into account the diffusion effects. Gupta et.al.[6] investigated the problem by taking into account the changes in density due to thermal and concentration effects and referred to the problem as the problem of generalized thermohaline (double-diffusive) shear flows.

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects[1,2]. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and this is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. Mohan[7] mollified the nastily behaving governing equations of Dufour-driven thermosolutal convection of the Veronis[8] type by the construction of an appropriate linear transformation and extended the result of Banerjee et. al.[9] concerning the linear growth rate and the behavior of oscillatory motion. Mohan et.al.[10] investigated the problem of Dufour-driven thermosolutal convection of the type described by Veronis in the completely confined fluid and derived a semi-circle theorem for the problem.

In the present paper we investigate the problem of Dufour-driven generalized double-diffusive shear flows. Sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system for the cases when the temperature and the concentration make opposing contributions to the vertical density gradient. The problem is completely solved at the marginal state when the basic velocity profile is linear and the diffusion of the dissolved solute is almost comparable than the diffusion of heat. A first approximation to the solution shows that as the initial density distribution increases, the Rayleigh number must also increase, a result which one would expect on physical grounds also.

THE PHYSICAL PROBLEM AND THE GOVERNING STABILITY EQUATIONS

An infinite horizontal layer of an initially stratified inviscid incompressible, heat conducting fluid with a differential streaming U(z), in the horizontal direction and density variation $\rho = \rho_0 f(z)$ in the vertical direction is confined between two horizontal boundaries $z = 0$ and $z = d$ maintained at constant temperatures T_0 and T_1 and solute concentrations S_0 and S_1 at the lower and upper boundaries respectively, where either $T_0 > T_1$, $S_0 > S_1$ or $T_0 < T_1$, $S_0 < S_1$, ρ_0 is a positive constant having the dimensions of density and f(z) is a monotonically decreasing function of the vertical coordinate z with a normalizing conditions f(0) = 1 without any loss of generality. The problem is to investigate the stability of this initial stationary state. Throughout our mathematical analysis we assume that f(z) and U(z) are respectively once and twice continuously differentiable everywhere in the flow domain.

Let the origin be taken on the lower boundary z = 0 with the z-axis perpendicular to it along the vertically upward direction so that the xy-plane then constitutes the horizontal plane z = 0. Then, the basic equations governing the problem in the framework of Boussinesq approximation are:

$$\frac{\partial Ui}{\partial t} + u_j \frac{\partial Ui}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \left[f(z) + \frac{\Delta\rho}{\rho_0} + \frac{\Delta\rho'}{\rho_0} \right] X_i \quad (1)$$

$$\frac{\partial Uj}{\partial x_j} = 0 \quad (2)$$

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0 \quad (3)$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = K_T \nabla^2 T + \gamma_1 \nabla^2 S \quad (4)$$

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = K_S \nabla^2 S \quad (5)$$

and $$\rho = \rho_0 [f(z) - \alpha(T - T_0) + \alpha'(S - S_0)] \quad (6)$$

In the above equations, $U_i (i = 1,2,3)$ are respectively the x, y and z components of velocity, p is the pressure, X_i represents the ith component of the external force, ρ is the density, K_T is the thermal diffusivity, K_S is the mass diffusivity, $\gamma_1 = \frac{D_{01}}{c_v}$ is the Dufour coefficient, α is the thermal coefficient of volume expansion and α' is the analogous solvent coefficient, $\Delta\rho = \rho_0\alpha(T_0 - T)$ and $\Delta\rho' = \rho_0\alpha'(S - S_0)$.

The initial stationary state whose stability we wish to examine is easily seen to be characterized by the following equations:

$$\left.\begin{aligned} &\mathbf{U_i = (u,v,w) = (U(z),0,0),}\\ &\mathbf{T = T_0 - \beta z,}\\ &\mathbf{S = S_0 - \beta' z,}\\ &\mathbf{\rho = \rho_0[f(z) + \alpha\beta z - \alpha'\beta' z],}\\ \textbf{and}\\ &\mathbf{p = -fg\rho dz} \end{aligned}\right\} \qquad (7)$$

where $\beta = \frac{T_0 - T_1}{d}$ and $\beta' = \frac{S_0 - S_1}{d}$ are the maintained uniform temperature and solute concentration gradient respectively.

Now following the usual steps of linear stability theory, we obtain the following linearized perturbation equations:

$$\mathbf{(n + Uk_x)(D^2 - k^2)W - k_x(D^2U)W}$$

$$\mathbf{= gk^2\left[\frac{w\frac{df}{dz}}{(n + Uk_x)} + i(\alpha\theta - \alpha'\phi)\right]} \qquad (8)$$

$$\mathbf{i(n + Uk_x)\theta - \beta W = K_T(D^2 - k^2)\theta + \gamma_1(D^2 - k^2)\phi} \qquad (9)$$

$$\mathbf{i(n + Uk_x)\phi - \beta' W = K_S(D^2 - k^2)\theta} \qquad (10)$$

where $D = \frac{d}{dz}$ W is the vertical component of the perturbed velocity, θ is the perturbed temperature, ϕ is the perturbed concentration, $k^2 = k_x^2 + k_y^2$ is the square of the wave number and n, is a constant which is complex in general.

In the subsequent discussions we shall take $k = k_x$, so that the investigation is confined to two dimensional motions. Using the non-dimensional quantities defined by

$$\mathbf{U^* = \frac{Ud}{K_T};\quad \sigma = \frac{ind^2}{K_T};\quad a = kd;\quad D = d\frac{d}{dz};}$$

$$\mathbf{R_1 = \frac{g\alpha\beta d^4}{K_T^2}; R_2 = \frac{g\left(\frac{df}{dz}\right)d^4}{K_T^2}; R_S = \frac{g\alpha'\beta' d^4}{K_T^2};}$$

$$\mathbf{W^* = \frac{gd^2}{K_T}W;\quad \theta^* = \theta;\quad \phi^* = \frac{\beta}{\beta'}\phi;\quad \tau' = \frac{K_S}{K_T};}$$

and dropping the asterisk for convenience, it follows from equations (8) – (10) that the non-dimensional linear stability equations governing the problem are given by

$$(U - C)^2(D^2 - a^2)W - (U - C)(D^2U)W +$$
$$R_3 W = iR_1 a(U - C)\theta - iR_S a(U - C)\phi \qquad (11)$$

$$[D^2 - a^2 - ia(U - C)]\theta + \gamma R_0(D^2 - a^2)\phi = -W \qquad (12)$$

$$\left[D^2 - a^2 - \frac{ia}{\tau}(U - C)\right]\phi = -\frac{W}{\tau} \qquad (13)$$

where $C = \frac{i\sigma}{a}, R_3 = \hat{R}_2 N^2, \hat{R}_2 = \frac{\rho_0 d^4}{K_T^2}, \gamma = \frac{\gamma_1}{K_T}$ is the Dufour number, $R_0 = \frac{\beta'}{\beta}$ and $N = -\frac{g\frac{df}{dz}}{\rho_0}$ is the Brunt-Vaisala frequency.

The solution of the equations (11)-(13) must be sought subject to the following boundary conditions:

$$W = 0 = \theta = \phi \quad at \quad z = 0 \quad and \quad z = 1 \qquad (14)$$

Equations (11) – (13) together with the boundary conditions (14) present an eigenvalue problem for $C(= C_r + iC_i)$, for given values of the other parameters and a given state of the system is stable, neutral or unstable provided C_i is negative, zero or positive respectively. Further, if $C_i = 0$ implies that $C_r = 0$ for every wave number a, then the principle of exchange of stabilities (PES) is valid, otherwise we have overstability at least when instability sets in a certain modes. It is to be noted that the inclusion of the convective effects of heat and mass transfer make the definitions of stable, neutral and unstable modes distinctly clear in the sense that the existence of a stable mode no longer implies the existence of an unstable mode etc., as is there in the classical instability problem of heterogeneous shear flows.

MATHEMATICAL ANALYSIS

We prove the following theorems

Theorem 1

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (11) – (14) with $R_1 > 0, R_S > 0, \gamma > 0, 0 < \tau < 1$ and

(i) $UD^2U > 0, \quad \forall z \in [0,1]$,

(ii) $R_S \le \left[\frac{UD^2U}{2} - R_3\right]_{\min} - \frac{R_1 R_0 \gamma}{1-\tau}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof

Using the transformation

$$\left.\begin{aligned} \tilde{\theta} &= \frac{1-\tau}{R_0\gamma}\theta + \phi \\ \tilde{\phi} &= \phi \\ \tilde{W} &= W \end{aligned}\right\} \quad (15)$$

the system of equations (11)-(13) together with the boundary condition (14)

assume the following forms:

$$(U-C)^2(D^2-a^2)W - (U-C)(D^2U)W + R_3W$$

$$= iR_1'a(U-C)\theta - iR_S'a(U-C)\phi \quad (16)$$

$$[D^2 - a^2 - ia(U-C)]\theta = -MW \quad (17)$$

$$\left[D^2 - a^2 - \frac{ia}{\tau}(U-C)\right]\phi = -\frac{W}{\tau} \quad (18)$$

With

$$W = 0 = \theta = \phi \quad at\ z = 0\, and\ z = 1 \quad (19)$$

where

$R' = \frac{R_1 R_0 \gamma}{1-\tau}$ (*Modified thermal Rayliegh number*),

$R_S' = R_S + R'$(*Modified concentration Rayliegh number*).

The sign tilde (~) has been omitted for simplicity.

If possible, let

$C_i = 0 \Rightarrow C_r = 0, \forall a$,

so that C = 0 is allowed by the governing equations and boundary conditions.

Equations (16) – (18) then assume the form

$$U^2(D^2-a^2)W - U(D^2U)W +$$

$$R_3W = iR_1'aU\theta - iR_S'aU\phi \quad (20)$$

$$(D^2 - a^2 - iaU)\theta = -MW \quad (21)$$

$$\text{and}\left(D^2 - a^2 - \frac{iaU}{\tau}\right)\phi = -\frac{W}{\tau} \quad (22)$$

In view of condition (i) of the theorem,

$U \neq 0, \quad \forall z \in [0,1]$,

so that equation (20) can also be written as

$$U(D^2-a^2)W - (D^2U)W$$

$$+\frac{R_3}{U}W = iR_1'aU\theta - iR_S'aU\phi \quad (23)$$

Multiplying equations (23), (21) and (22) by $W^*, \frac{-iR_1'a\theta^*}{M}$ and $i\tau R_S' a\phi^*$ (*indicates, complex conjugation) respectively, integrating over the vertical range of z by parts appropriately, using the boundary conditions (19) and adding the resulting equations we get

$$\int_0^1 U(|DW|^2 + a^2 dz + \int_0^1 W^* U\ DWdz + \int_0^1 D^2U\,|$$

$$W|^2\, dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2\, dz +$$

$$i\tau R_S' a\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz - \frac{iR'a_1}{M}\int_0^1 (|D\theta|^2 + a^2|$$

$$\theta|^2)dz = R_S' a^2\int_0^1 U|\phi|^2\, dz +$$

$$\int_0^1 \frac{R_3}{U}|W|^2\, dz + 2iR_S' a\,\mathrm{Re}\left(\int_0^1 \phi W^* dz\right) -$$

$$2iR_1' a\,\mathrm{Re}\left(\int_0^1 \theta W^* dz\right) \quad (24)$$

where ‘Re’ stands for the real part.

Integrating the second term on the left hand side of equation (24) by parts once and using boundary conditions (19), we get

$$\mathbf{Re}\left(\int_0^1 \mathbf{W^*DUDWdz}\right) = \frac{-1}{2}\int_0^1 \mathbf{D^2U\,|\,W\,|^2\,dz.} \qquad \mathbf{(25)}$$

Equating the real parts of equation (24), we have upon using equation (25) that

$$\int_0^1 U(|DW|^2 + a^2|W|^2)dz + \frac{1}{2}\int_0^1 D^2U\,|W|^2\,dz + \frac{R_1'}{M}\,a^2\int_0^1 U\,|\theta|^2\,dz = \int_0^1 \frac{R_3}{U}|W|^2\,dz + R_S' a^2\int_0^1 U\,|\phi|^2\,dz \qquad \mathbf{(26)}$$

Multiplying equation (22) by its complex conjugate and integrating over the vertical range of z, we get

$$\int_0^1 \frac{1}{U}|(D^2 - a^2)\phi|^2\,dz + \frac{a^2}{\tau^2}\int_0^1 U\,|\phi|^2\,dz = \frac{1}{\tau^2}\int_0^1 \frac{1}{U}|W|^2\,dz \qquad \mathbf{(27)}$$

Condition (i) of the theorem implies that either (a) $U > 0, D^2U > 0$ or (b) $U < 0, D^2U < 0, \quad \forall z \in [0,1]$.

If (a) holds, then equation (27) gives

$$\int_0^1 U\,|\phi|^2\,dz < \frac{1}{a^2}\int_0^1 \frac{1}{U}|W|^2\,dz \qquad \mathbf{(28)}$$

Using inequality (28) in equation (26), we get

$$\int_0^1 U(|DW|^2 + a^2|W|^2)dz + \frac{R_1'a^2}{M}\int_0^1 U\,|\theta|^2\,dz. + \int_0^1 \frac{1}{U}\left(\frac{UD^2U}{2} - R_3 - R_S'\right)|W|^2\,dz < 0 \qquad \mathbf{(29)}$$

If (b) holds, it is easily seen that inequality (29) assumes the form

$$\int_0^1 |U|(|DW|^2 + a^2|W|^2)dz + \frac{R_1'}{M}a^2\int_0^1 |U||\theta|^2\,dz + \int_0^1 \frac{1}{|U|} - \left(\frac{UD^2U}{2} - R_3 - R_S'\right)|W|^2\,dz < 0 \qquad \mathbf{(30)}$$

Inequalities (29) – (30) obviously cannot hold under conditions (i) and (ii) of the theorem $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

This completes the proof of the theorem.

The essential content of *Theorem 1*, from the point of view of hydrodynamic instability is that an arbitrary neutral mode in the problem of Dufour-driven generalized thermohaline shear flows of Veronis'[7] $(R_1 > 0, R_S > 0)$ is definitely not non-oscillatory $(C_r = 0)$ in character, i.e., PES is not valid if $UD^2U > 0$, everywhere in [0,1] and $R_S' \le \left[\frac{UD^2U}{2} - R_3\right]_{\min}$ or $R_S \le \left[\frac{UD^2U}{2} - R_3\right]_{\min} - \frac{R_1R_0\gamma}{1-\tau}$

Special cases

If follows from *Theorem 1*, that the PES is not valid for Dufour-driven thermal shear flows $(R_S = 0 = R_3)$ if $UD^2U > 0, \quad \forall z \in [0,1]$ and $UD^2U > \frac{2R_1R_0\gamma}{1-\tau}, \quad \forall z \in [0,1]$.

In fact if U is linear and $U(z) \neq 0, \quad \forall z \in [0,1]$, then one could see from the proof of the theorem that the result remains valid.

(ii) Dufour-driven generalized thermal shear flows $(R_S = 0)$ if $UD^2U > 0, \quad \forall z \in [0,1]$ and $\left[\frac{UD^2U}{2} - R_3\right]_{\min} > \frac{R_1R_0\gamma}{1-\tau}$.

(iii) Dufour-driven thermohaline shear flows of Veronis' type $(R_3 = 0, R_1 > 0, R_S > 0)$ if $UD^2U > 0, \quad \forall z \in [0,1]$ and $R_S \le \left(\frac{UD^2U}{2}\right)_{\min} - \frac{R_1R_0\gamma}{1-\tau}$.

Theorem 2

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (16) – (19) with $R_1 > 0, R_S > 0$ and

(i) (a) $U > 0$ and $D^2U \le 0, \quad \forall z \in [0,1]$,

(b) $U < 0$ and $D^2U \ge 0, \quad \forall z \in [0,1]$,

(ii) $R_S \le \left[\Pi^2U^2 - \frac{1}{2}|UD^2U| - R_3\right]_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

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Proof

If possible, let

$C_i = 0 \Rightarrow C_r = 0, \forall a$,

so that C = 0 is allowed by the governing equations and boundary conditions.

Proceeding exactly as in Theorem 1 upon considering the case i(a), we have

$$\int_0^1 U(|DW|^2 + a^2|W|^2)dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2 dz = \int_0^1 \frac{R_3}{U}|W|^2 dz + R_S'a^2\int_0^1 U|\phi|^2 dz + \frac{1}{2}\int_0^1 |D^2U|(W)^2 dz \quad \textbf{(31)}$$

Equation (31) together with inequality (28), yields

$$\int_0^1 U(|DW|^2 + a^2|W|^2)dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2 dz < \int_0^1 \frac{R_3}{U}|W|^2 dz + R_S'a^2\int_0^1 \frac{1}{U}|W|^2 dz + \frac{1}{2}\int_0^1 |D^2U||W|^2 dz \quad \textbf{(32)}$$

Now, since $U > 0, \quad \forall z \in [0,1]$, we have

$$\int_0^1 U|DW|^2 \geq U_{\min}\int_0^1 |DW|^2 dz,$$

which upon using the Poincare inequality, namely,

$$\int_0^1 |Df_1|^2 \geq \Pi^2\int_0^1 |f_1|^2 dz \quad \textbf{(33)}$$

where $f_1(0) = 0 = f_1(1)$ with $f_1 = W$, gives

$$\int_0^1 U|DW|^2 dz \geq \Pi^2 U_{\min}\int_0^1 |w|^2 dz \quad \textbf{(34)}$$

Using inequality (34) in the inequality (32), we get

$$\int_0^1 \frac{1}{U}\left\{(\Pi^2 U_{\min})U - \frac{1}{2}|UD^2U| - R_3 - R_S'\right\}|W|^2 dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2 dz < 0 \quad \textbf{(35)}$$

Similarly in case i(b), it is easily seen that inequality (35) assumes the form

$$\int_0^1 \frac{1}{|U|}\left\{(\Pi^2 |U|_{\min})|U| - \frac{1}{2}|UD^2U| - R_3 - R_S'\right\}|W|^2 dz + \frac{R_1'}{M}a^2\int_0^1 |U||\theta|^2 dz < 0 \quad \textbf{(36)}$$

Inequalities (35) – (36) obviously cannot hold under the conditions of the theorem.

Hence, under the conditions of the theorem

$C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

This completes the proof of the theorem.

The essential content of Theorem 2, from the point of view of hydrodynamic instability is similar to that of Theorem 1.

Special cases

If follows from Theorem 2, that the PES is not valid for

(i) Dufour-driven thermal shear flows if $\mathbf{U > 0, \quad D^2U \leq 0, or \quad U < 0, D^2U \geq 0, \quad \forall z \in [0,1]}$

and $\left(\Pi^2U^2 - \frac{1}{2}|UD^2U| - \frac{R_1R_0\gamma}{1-\tau}\right)_{\min} \geq 0$.

(ii) Dufour-driven generalized thermal shear flows if $U > 0, \quad D^2U \leq 0, \quad \text{or} \quad U < 0, D^2U \geq 0, \quad \forall z \in [0,1]$

and $\left(\Pi^2U^2 - \frac{1}{2}|UD^2U| - R_3 - \frac{R_1R_0\gamma}{1-\tau}\right)_{\min} \geq 0$.

(iii) Dufour–driven thermohaline shear flows of Veronis' type if $U > 0, \quad D^2U \leq 0, \quad \text{or} \quad U < 0, D^2U \geq 0, \quad \forall z \in [0,1]$

and $R_S' \leq \left(\Pi^2U^2 - \frac{1}{2}|UD^2U|\right)_{\min}$ or

$R_S \leq \left(\Pi^2U^2 - \frac{1}{2}|UD^2U|\right)_{\min} - \frac{R_1R_0\gamma}{1-\tau}$

Theorem 3

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (16) – (19) with $R_1 < 0, R_S < 0$ and

(i) $UD^2U > 0, \quad \forall z \in [0,1]$,

(ii) $|R_1| \le \frac{1-\tau}{M^2 R_0 \gamma}\left(\frac{UD^2U}{2} - R_3\right)_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof

Putting

$R_1 = -|R_1|$ and $R_S = -|R_S|$

in equation (26), and using the inequality

$$\int_0^1 U|\theta|^2\, dz < \frac{M^2}{a^2}\int_0^1 \frac{1}{U}|W|^2\, dz \qquad \textbf{(37)}$$

which is derived from equation (17) in a manner similar to the derivation of inequality (28), and proceeding exactly as in Theorem 1, we get the result.

This completes the proof of theorem.

Theorem 4

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (16) – (19) with $R_1 < 0, R_S < 0$ and

(i) (a) $U > 0,\ D^2U \le 0,\quad \forall z \in [0,1]$, or

(b) $U < 0,\ D^2U \ge 0,\quad \forall z \in [0,1]$,

(ii) $|R_1| \le \frac{1-\tau}{M^2 R_0 \gamma}\left(\Pi^2 U^2 - \frac{1}{2}|UD^2U| - R_3\right)_{\min}$,

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof

Putting $R_1 = -|R_1|$ and $R_S = -|R_S|$ in equation (26), using inequalities (34) and (37) and proceeding exactly as in Theorem 2, we get the result.

This completes the proof of the theorem.

The essential contents of Theorem 3 and 4, from the point of view of hydrodynamic instability are similar to the two earlier theorems. However, presently the problem is that of generalized thermohaline shear flows of Stern's [11] type $(R_1 < 0, R_S < 0)$. Further, special cases of Theorems 3 and 4 analogous to that of the earlier theorems could be easily written down in the present case also.

Theorem 5

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (16) – (19) with $R_1 > 0, R_S > 0$, then

$$C_i < \frac{q + \sqrt{(q^2 + 4R_1'M)}}{2\pi},$$

Where $q = (|D^2U|)^{\max}$.

Proof

Since $U - C \neq 0, \forall z \in [0,1]$, therefore dividing equation (16) throughout by (U – C) and then proceeding as in Theorem 1, we get

$$\int_0^1 (U-C)(|DW|^2 + a^2|W|^2)dz + \int_0^1 (DU)W^* DWdz + \int_0^1 D^2U|W|^2\, dz - \frac{R_1'}{M}a^2$$

$$\int_0^1 (U-C)|\theta|^2\, dz + i\tau R_S a\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz - \frac{iR_1'a}{M}\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz =$$

$$= R_S'a^2\int_0^1 (U-C)|\phi|^2\, dz + \int_0^1 \frac{R_3}{(U-C)}|W|^2\, dz + 2iR_S'a\,\mathrm{Re}\left(\int_0^1 \phi WD^* dz\right) - iR_1'a\,\mathrm{Re}\left(\int_0^1 \phi WD^* dz\right) \qquad \textbf{(38)}$$

Equating the imaginary parts of equation (38) and dividing the resulting equation throughout by $C_i (> 0)$, we get

$$\int_0^1 (|DW|^2 + a^2|W|^2)dz + \frac{R_1'}{M}a^2\int_0^1 |\theta|^2\, dz + \frac{R_1'a}{C_iM}\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz + \int_0^1 \frac{R_3}{|U-C|^2}|W|^2\, dz - \frac{2R_S'a}{C_i}\mathrm{Re}\left(\int_0^1 \phi W^* dz\right) = \mathrm{Im}\left(\frac{1}{C_i}\int_0^1 (DU)W^* dz\right) + \frac{\tau R_S'a}{C_i}\int_0^1 (|D\phi|^2 + a^2|\phi|^2\, dz +$$

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$$+R'_S a^2\int_0^1 |\phi|^2\,dz+\frac{2R'_1 a}{C_i}\mathrm{Re}\left(\int_0^1 \theta W^* dz\right) \quad \textbf{(39)}$$

where 'Im' stands for the imaginary part.

Using equations (17) – (18), it follows that

$$\mathrm{Re}\left(\int_0^1 \theta W^* dz\right)=\frac{1}{M}\int_0^1(|D\theta|^2+a^2|\theta|^2+\frac{1}{M}aC_i\int_0^1|\theta|^2\,dz \quad \textbf{(40)}$$

and

$$\mathrm{Re}\left(\int_0^1 \phi\, W^* dz\right)=\tau\int_0^1(|D\phi|^2+a^2|\varphi|^2+aC_i\int_0^1|\phi|^2\,dz \quad \textbf{(41)}$$

Substituting from equations (40) – (41) in equation (39) and simplifying the resulting equation, we get

$$\int_0^1(|DW|^2+a^2|W|^2)dz+\frac{\tau R'_S a}{C_i}\int_0^1(|D\phi|^2+a^2|\phi|^2+\frac{aC_i}{\tau}|\phi|^2)dz+$$

$$+\int_0^1\frac{R_3}{|U-C|^2}|W|^2\,dz=\frac{R'_1 a}{MC_i}\int_0^1(|D\theta|^2+a^2|\theta|^2+aC_i|\theta|^2)dz+$$

$$+\mathrm{Im}\left(\frac{1}{C_i}\int_0^1(\mathrm{DU})\mathrm{W}^*\mathrm{DWdz}\right) \quad \textbf{(42)}$$

Multiplying equation (17) by θ^*, integrating over the vertical range of z by parts once, using the boundary conditions (19) and equating the real parts of the resulting equation, we get

$$\frac{1}{M}\int_0^1(|D\theta|^2+a^2|\theta|^2+aC_i|\theta|^2)dz=\mathrm{Re}\left(\int_0^1 W\theta^* dz\right)$$

$$\le\left|\int_0^1 \mathrm{W}\theta^*\mathrm{dz}\right|$$

$$\le\int_0^1|\mathrm{W}||\theta|\,\mathrm{dz}$$

$$\le\left\{\int_0^1|\mathbf{W}|^2\mathbf{dz}\right\}^{1/2}\left\{\int_0^1|\boldsymbol{\theta}|^2\mathbf{dz}\right\}^{1/2} \quad \textbf{(43)}$$

It follows from inequality (42) that

$$\frac{1}{M}aC_i\int_0^1|\theta|^2\,dz<\left\{\int_0^1|W|^2dz\right\}^{1/2}\left\{\int_0^1|\theta|^2dz\right\}^{1/2}$$

i.e. $\left\{\int_0^1|\theta|^2dz\right\}^{1/2}<\frac{M}{aC_i}\left\{\int_0^1|W|^2dz\right\}^{1/2}$ since,

$$a>0, C_i>0 \quad \textbf{(44)}$$

Using inequality (43) in inequality (42), we get $\frac{1}{M}\int_0^1(|D\theta|^2+a^2|\theta|^2+aC_i|\theta|^2)dz<\frac{M}{aC_i}\int_0^1|W|^2\,dz$, which upon using inequality (33) with $f_1=W$, gives

$$\int_0^1(|D\theta|^2+a^2|\theta|^2+aC_i|\theta|^2)dz<\frac{M^2}{aC_i\pi^2}\int_0^1|DW|^2\,dz \quad \textbf{(45)}$$

Further,

$$\mathrm{Im}\left(\frac{1}{C_i}\int_0^1(\mathrm{DU})\mathrm{W}^*\mathrm{DWdz}\right)\le\frac{1}{aC_i}\int_0^1|\mathrm{DU}||\mathrm{W}||\mathrm{DW}|\,\mathrm{dz},$$

$$\le\frac{q}{aC_i}\int_0^1|\mathrm{W}||\mathrm{DW}|\,\mathrm{dz}$$

$$\le\frac{q}{aC_i}\left\{\int_0^1|\mathrm{W}|^2\,\mathrm{dz}\right\}^{1/2}\left\{\int_0^1|\mathrm{DW}|^2\,\mathrm{dz}\right\}^{1/2}$$

which upon using inequality (33) with $f=W$, gives

$$\mathbf{Im}\left(\frac{1}{\mathbf{C_i}}\int_0^1(\mathbf{DU})\mathbf{W}^*\mathbf{DWdz}\right)\le\frac{\mathbf{q}}{\pi\mathbf{C_i}}\int_0^1|\mathbf{DW}|^2\,\mathbf{dz} \quad \textbf{(46)}$$

Where $q=(|DU|)_{max}$.

Equation (42) upon using inequalities (44) – (45), gives

$$\left[1-\frac{R'_1 M}{\pi^2 C_i^2}-\frac{q}{\pi C_i}\right]\int_0^1|DW|^2\,dz+a^2\int_0^1|W|^2\,dz+$$

$$+\frac{\tau R'_S a}{C_i}\int_0^1\left(|D\phi|^2+a^2|\phi|^2\frac{aC_i}{\tau}|\phi|^2\right)$$

$$dz+\int_0^1\frac{R_3}{|U-C|^2}|W|^2\,dz<0 \quad \textbf{(47)}$$

Since $a>0,\ C_i>0,\ \tau>0,\ R'_S>0\ and\ R_3>0$,

Therefore inequality (46) clearly implies that $\pi^2C_i^2-\pi qC_i-R'_1M<0$.

Hence,

$$C_i<\frac{q+\sqrt{(q^2+4R'_1M)}}{2\pi}.$$

This completes the proof of the theorem.

The essential content of Theorem 5 from the point of view of hydrodynamic instability is that the growth rate of an arbitrary unstable $(C_i > 0)$ mode in the problem of Dufour-driven generalized thermohaline shear flows of Veronis' type $(R_1 > 0, R_S > 0)$ is necessarily bounded with upper bound $\frac{q+\sqrt{(q^2+4R_1'M)}}{2\pi}$. Further, this result is uniformly valid for the problems of thermal shear flows, generalized thermal shear flows and thermohaline shear flows of Veronis' type.

Theorem 6

If $(C, W, \theta, \phi), C = C_r + iC_i$ is a solution of equations (16) – (19) with $R_1 < 0, R_S < 0$, then

$$C_i < \frac{q+\sqrt{(q^2+4|R_S'|)}}{2\pi}$$

Where q is as defined in Theorem 5.

Proof

Putting $R_1 = -|R_1|$ and $R_S = -|R_S|$ in equation (42), using the inequalities (45) and

$$\tau\int_0^1 (|D\phi|^2 + a^2|\phi|^2)dz + aC_i\int_0^1 |\phi|^2,$$

$$dz < \frac{1}{aC_i\pi^2}\int_0^1 |DW|^2\, dz \quad \textbf{(48)}$$

which is derived from equation (18) in a manner similar to the derivation of inequality (44), we get the result.

This completes the proof of the theorem.

The essential content of *Theorem 6* from the point of view of hydrodynamic instability is similar to that of Theorem 5. However, the problem presently is that of Dufour-driven generalized thermohaline shear flows of Stern's type $(R_1 < 0, R_S < 0)$.

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