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ON ASYMPTOTIC EXPANSION OF GENERALIZED MELLIN-WHITTAKER TRANSFORM

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ABSTRACT

In this article, we investigate the asymptotic behaviour at infinity of the generalized Mellin-Whittaker transform of the form.

$$MW_{\downarrow}(k,m)^{\uparrow}\rho\ f](s,y) = \int_{\downarrow} 0^{\uparrow}\infty \left[\left(\int_{\downarrow} 0^{\uparrow}\infty\ x^{\uparrow}(s-1)e^{\uparrow}(-yt/(2))\right)\left(\left[(yt)\right]^{\uparrow}\rho\right)\right] W_{\downarrow}(k,m)\,(yt)\,f\,(x,t)\,dxd$$

Involving the Whittaker function $W_{k,m}(z)$ inthekernel. It is proved that $[MW_{\downarrow}(k,m)^{\uparrow}\rho\ f](s,y)$ has power or power logarithmic asymptotic expansion as $s \to \infty$ and $y \to \infty$ provided that $f(x,t)$ has power asymptotic behaviour at infinity.

Key words: Asymptotic expansion, Mellin-Whittaker transform.

INTRODUCTION

Extension of some transformations to generalized functions have been done from time to time and their properties have been studied by various mathematicians. Number of double transformations have been developed by many mathematicians like Fourier-Hankel transformation[1], Mellin-Whittaker transform[6] etc. In[7] and[8] we have investigated the properties and inversion for Mellin-Whittaker transform and fractional Mellin-Whittaker transform.

Now in this article we have investigated the asymptotic behaviour at infinity of the generalised Mellin-Whittaker transform for the special case p = 1, q = 1 and $\rho = m - \frac{1}{2}$ and of the form –

$$\left[MW_{k,m}^{\rho} f\right](s,y) = \int_0^{\infty}\int_0^{\infty} f(x,t) K\,(x,t,s,y)\,dxdt, \qquad \ldots(1.1)$$

where the kernel $K_{k,m}^{\rho}(x,t,s,y) = x^{s-1}e^{-(yt/2)(yt)^{\rho}W_{k,m}(yt)\,f(x,t)\,dxdt}$

On asymptotic expansion of generalized mellin-whittaker transform $MW_{k,m}^{\rho}$ at infinity

Using the asymptotic behaviour of the Kummer and Tricomi functions[2] at infinity, we have the following asymptotic estimate of the Mellin-Whittaker function, for any fixed $x > 0$.

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$$K(x,t,s,y) = 0\left[(x\right]\uparrow s\, e\uparrow\left((-t)/2\right)t\uparrow Rek\big) \text{ as } s \to \infty \text{ and } t \to \infty$$

In a series of papers[3-5] Handelsman and Lew have developed a theory which yields asymptotic expansion of integrals of the form.

$$I(\lambda) = \int_0^\infty \left[g(\xi)\, f(\lambda\xi)\right] d\xi, \text{ when } \lambda \to \infty$$

For our purpose we take integrals of the form –

$$I(s,y) = \int_0^\infty \int_0^\infty g(\eta,\xi)\, f\,(\lambda\eta, y\xi)\, d\eta d\xi, \qquad \ldots(2.1)$$

when $s \to \infty$ and $y \to \infty$

Mellin-Whittaker transform as per equation (1.1) is –

$$\left[MW_{k,m}^{\rho} f\right](s,y) = \int_0^\infty \int_0^\infty x^{s-1} e^{-(yt/2)(yt)^{\rho}}\, W_{k,m}(yt)\, f(x,t)\, dxdt \qquad \ldots(2.2)$$

If we set $x = \lambda\eta$ and $t = y\xi$ in (2.2) then we obtain –

$$[MW]_{k,m}^{\rho} f\big](s,y) = \frac{1}{2\pi i}\lambda^{s}\, y^{2\rho+1} \int_0^\infty \int_0^\infty \eta^{s-1}\, \xi^{\rho}\, e^{-\left(y^{z\xi}/2\right)} W_{k,m}(y^2\xi)\, f(\lambda\eta, y\xi)\, d\eta d\xi \qquad \ldots(2.3)$$

This integral is given by (2.1) with

$$g(\eta,\xi) = \lambda^{s}\eta^{s-1}\, y(y^2\xi)^{\rho}\, e^{-\left(y^{z\xi}/2\right)} W_{k,m}(y^2\xi)$$

Hence we can apply the method developed in[5].

We first note that under suitable conditions on $f\,(x,t)$ as per[9] the Parseval relation for Mellin transform can be used.

$$MW_{\downarrow}(k,m)^{\uparrow}\rho\; f\,](s,y) = 1/2\pi i\, \lambda^{\uparrow}s \int_{\downarrow} 0^{\uparrow}\infty \int_{\downarrow}(c-i\infty)^{\uparrow}(c+i\infty)\;\; \eta^{\uparrow}(s-1)\;\; y^{\uparrow}(-z)\, M[f,z]\, M\left[g,1-z\right] f$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-z} M\,[f,s]\, M\,[f,z]\, M\,[g,1-z]\, dz$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-z} M\,[f,s]\, M\,[f,z]\, M\,[g,1-z]\, dz$$

has the form,

$$[[W]_{k,m}^{\rho} f\big](s,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{z-1} M\,[f,s]\, M\,[f,z]\, \frac{\Gamma\left(\rho + m + {}^{3}\!/_{2-z}\, \Gamma\left(\rho - m + {}^{3}\!/_{2-z}\right)\right)}{\Gamma\left(\rho - k + 2 + z\right)}\, dz \qquad \ldots(2.4)$$

for the transform (2.3). Here

$$M[g,z] = y^{1-2z}\, \frac{\Gamma\left(\rho + m + 1\!/_{2+z}\, \Gamma\left(\rho - m + 1\!/_{2+z}\right)\right)}{\Gamma\left(\rho - k + 2 + z\right)} \qquad \ldots(2.5)$$

represents the Mellin transform of $g(t)$ evaluated at z, and the contour of integration is a Bromwich contour in the common domain of analyticity of $M [f, s]$, $M [f, z]$, $M [g, -z]$ and for simplicity,we assume that $M [f, z]$ exist in the ordinary sense. In the case $s \to \infty$ and $y \to \infty$ the desired expansion is obtained by displacing the contour in equation (2.4) to the left by using the fact that $M [f, s]$ converge absolutely in $-a < Res < -b$ where a and b are the real numbers given by –

$a = \sup \{r : f(x) = 0\,(x^u)\ as\ x \to 0+\}$

$a = \inf \{s : f(x) = 0\,(x^v)\ as\ x \to 0+\infty\}$

and $M [f, s]$ and $\dfrac{\Gamma\left(\rho + m + {}^{3}\!/_{2-z}\,\Gamma\left(\rho - m + {}^{3}\!/_{2-z}\right)\right)}{\Gamma\left(\rho - k + 2 - z\right)}$ is analytic in the half plane

$Res < -b$ and $Rez < min\ \{\mathrm{Re}\left[(3/2 + \rho + m); \mathrm{Re}\left[(3/2 + \rho - m)\right)\right)\}$ respectively.

Thus to calculate the resdue series, we need only analytically continue $M [f, s]$ and $M [f, z]$ into the left half –

$Res < -b$ and $Rez < min\ \{\mathrm{Re}\left[(3/2 + \rho + m); \mathrm{Re}\left[(3/2 + \rho - m)\right)\right)\}$ as ameromorphic function :

$$f(x,t) \sim \sum_{i,j=0}^{\infty} A_i x^{a_i} B_j t^{b_j}\,, \quad Re\,(a_i), Re\,(b_j) \uparrow \infty \qquad \ldots(2.6)$$

Then $M [f, z]$ and $M [f, s]$ has simple pole at the points

$s = a_i$ and $z = -b_j$ and its Laurent expansion near these poles have singular parts

$$\frac{A_i}{s + a_i} \text{ and } \frac{B_j}{z + b_j} \qquad \ldots(2.7)$$

It follows from equations (2.4) and statement (2.7) and the assumption that we are justified in displacing the Bromwich contour in equation (2.4) arbitrarily far to the left and calculation of the residue series gives us the next result.

Theorem: Let function $f(x, t)$ be locally integrable on $(0, \infty)$ and satisfy equation (2.6). Then the integral transform $\left[MW_{k,m}^{\rho} f\right](s,y)$ has asymptotic expansion, as $s \to \infty$ and $y \to \infty$.

$$[MW_{k,m}^{\rho} f](s,y) \sim \sum_{i,j=0}^{\infty} Ais^{a_i} B_j y^{-b_j - 1} \frac{\Gamma\left(\rho + m + {}^{3}\!/_{2-bj}\,\Gamma\left(\rho - m + {}^{3}\!/_{2-bj}\right)\right)}{\Gamma\left(\rho - k + 2 - b_j\right)}$$

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