ISSN : 0974 - 7435 Volume 10 Issue 11

2014

BioTechnology

An Indian Journal

FULL PAPER

BTAIJ, 10(11), 2014 [5477-5483]

Moebius sectional curvature of conformal submanifolds on S^n

Nan Ji[1], Yuxia Tong[1], Kaiwen Guo[2]

[1]College of Sciences, Hebei United University, Tangshan 063009, Hebei, (CHINA)

[2]College of Sciences, Tianjin Polytechnic University, Tianjin 300387, (CHINA)

E-mail : pumpkinji@139.com

ABSTRACT

In this paper, let M^R be a n-dimensional submanifold without umbilical point on unit sphere S^R, we use some Moebius invariants to get two pinching theorems about the Moebius sectional curvature, which give the characterizations of Veronese submanifolds and Clifford tori.

KEYWORDS

Conformal submanifold; Moebius sectional curvature; Moebius invariants.

INTRODUCTION

Let M^n be a n-dimensional submanifold without umbilical point on unit sphere S^n, Wang (cf.[1])using conformal differential geometry to establish the theory of conformal differential geometry of submanifolds, and give the classification of the vanishing Moebius form in unit sphere S^3. Submanifolds are obtained fully invariant system under the conformal group. Many conformal submanifolds in differential geometry was classified completely (cf[2-7]), which apply the invariant system---Moebius form, Moebius second fundamental form B, Blaschake tensor A, and then submanifold of unit sphere S^n is given a number of important Moebius characters. In this paper, we prove two Moebius sectional curvature pinching theorems, which give the characterizations of Clifford tori and Veronese submanifolds by the Moebius invariants.

Orthonormal frame field and Riemannian curvature. Let M be a n-dimentional Riemannian manifold, $e_1, e_2, \cdots, e_n$ a local orthonormal frame field on M, and $\omega_1, \omega_2, \cdots, \omega_n$ is its dual frame field. Then the structure equation of M are given by:

$$\mathbf{d}\omega_{\mathbf{i}} = \sum_{\mathbf{j}} \omega_{\mathbf{j}} \wedge \omega_{\mathbf{ji}}, \ \omega_{\mathbf{ij}} = -\omega_{\mathbf{ji}} \quad (1)$$

$$\mathbf{d}\omega_{\mathbf{ij}} = \sum_{\mathbf{l}} \omega_{\mathbf{il}} \wedge \omega_{\mathbf{lj}} - \frac{1}{2}\sum_{\mathbf{k,l}} \mathbf{R}_{\mathbf{ijkl}}\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}} \quad (2)$$

where ω_{ij} is the Levi-civita connection and R_{ijkl} the Riemannian curvature tensor of M. Ricci tensor R_{ij} and scalar curvature are defined respectively by

$$R_{ij} := \sum_k R_{kikj}, r := \sum_k R_{kk} \quad (3)$$

MOEBIUS INVARIANTS

Let R_1^{n+2} is a n+2-dimentional Lorentzian space,

$$X = (x_0, x_1, ..., x_{n+1}), \ Y = (y_0, y_1, ..., y_{n+1}),$$

define $< X, Y > = -x_0 y_0 + x_1 y_1 + ... + x_{n+1} y_{n+1}$ **(4)**

Let $x : M^m \to S^n$ is an Immersed submanifolds without umbilical point on unit sphere, and position vector $Y = \rho(1, x)$,

$$\rho^2 = \frac{m}{m-1}(\|II\|^2 - mH^2) \quad (5)$$

then $g = < dY, dY > = \rho^2 dx \cdot dx$ is Moebius invariants.

In the unit sphere, let $\{e_1, e_2, \cdots, e_m\}$ a local orthonormal frame field on M, and $\{\omega_1, \omega_2, \cdots, \omega_m\}$ is its dual frame field. where $1 \le i, j, k, l..... \le m$; $m+1 \le \alpha, \beta, ... \le m + p = n$, then

$$\mathrm{A} = \sum_{i,j} A_{ij}\omega_i \wedge \omega_j, \ \mathrm{B} = \sum_{i,j,\alpha} B_{ij}^{\alpha}\omega_i \wedge \omega_j E_\alpha, \ \phi = \sum_{i,\alpha} C_i^\alpha \omega_i E_\alpha$$

where A is Blaschake tensor, B is Moebius form, ϕ is Moebius second fundamental form, then we get the equation as follows :

$$A_{ij,k} - A_{ik,j} = \sum_{\alpha} (B_{ik}^{\alpha} C_j^{\alpha} - B_{ij}^{\alpha} C_k^{\alpha}) \quad \textbf{(6)}$$

$$C_{i,j}^{\alpha} - C_{j,i}^{\alpha} = \sum_{k} (B_{ik}^{\alpha} A_{kj} - B_{kj}^{\alpha} A_{ki}) \quad \textbf{(7)}$$

$$B_{ij,k}^{\alpha} - B_{ik,j}^{\alpha} = \delta_{ij} C_k^{\alpha} - \delta_{ik} C_j^{\alpha} \quad \textbf{(8)}$$

$$R_{ijkl} = \sum_{\alpha} (B_{ik}^{\alpha} B_{jl}^{\alpha} - B_{il}^{\alpha} B_{jk}^{\alpha}) + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il} \quad \textbf{(9)}$$

$$\sum_i B_{ii}^{\alpha} = 0, \; trA = \frac{1+m^2 R}{2m}, \; \sum_{\alpha} \sum_{i,j} (B_{ij}^{\alpha})^2 = \frac{m-1}{m}, \; R = \frac{1}{m(m-1)} \sum_{i,j} R_{ijij} \quad \textbf{(10)}$$

where R_{ijkl} is the Riemannian curvature tensor of M, R is the normal Moebius scalar curvature of M.

PINCHING THEOREMS

Lemma 1 Let $x: M^m \to S$ is submainfolds without umbilical point on S^n, $\forall a \in R^1$

$$0 = \frac{1}{2}\Delta\|B\|^2 = \|\nabla B\|^2 + (1+a) \sum_{i,j,\alpha,\, t,k} B_{ij}^{\alpha} (B_{tk}^{\alpha} R_{tijk} + B_{ti}^{\alpha} R_{tkjk}) - \frac{(m-1)a}{m} trA$$

$$+ a\sum_{\alpha} tr(B_{\alpha} \nabla \phi_{\alpha}) - (1-a)\sum_{\alpha,\beta} [tr(B_{\alpha}^2 B_{\beta}^2) - tr(B_{\alpha} B_{\beta})^2] + a\sum_{\alpha,\beta} [tr(B_{\alpha} B_{\beta})]^2 - ma\sum_{\alpha} tr(AB_{\alpha}^2)$$ Lemma 2

$$2\sum_{\alpha,\beta} [tr(B_{\alpha}^2 B_{\beta}^2) - tr(B_{\alpha} B_{\beta})^2] + \sum_{\alpha,\beta} [tr(B_{\alpha} B_{\beta})]^2 \le [1 + \frac{1}{2}\mathrm{sgn}(p-1)]\|B\|^4 \quad \textbf{(11)}$$

if and only if

(i) $p = 1$

or (ii) $p = 2$, B^{m+1}, B^{m+2} at the same time as

$$\lambda \overline{B}^{m+1}, \mu \overline{B}^{m+2}, \lambda^2 = \mu^2,$$

$$\overline{B}^{m+1} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \; \overline{B}^{m+2} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$$

Theorem 1 Let $x:\mathrm{M}^{\mathrm{m}}\to\mathrm{S}^{\mathrm{n}}\ (n=m+p)$ is submainfold without umbilical point in S^n, K is the infimum of the sectional curvature, then

$$K\le\frac{m-1}{2m^2}[1+\frac{1}{2}\operatorname{sgn}(p-1)-\frac{1}{p}],$$

if $K\ge\frac{m-1}{2m^2}[1+\frac{1}{2}\operatorname{sgn}(p-1)-\frac{1}{p}]$,

$x(M)$ is Moebius equivalent to a Versonese surface in S^4,

or is equivalent to Clifford tori in S^{m+1}, $S^k(\sqrt{\frac{k}{m}})\times S^{m-k}(\sqrt{\frac{m-k}{m}})\ (1\le k\le m-1)$

Theorem 2 Let $x:\mathrm{M}^{\mathrm{m}}\to\mathrm{S}^{\mathrm{n}}\ (n=m+p)$ is submainfolds without umbilical point in S^n, K is infimum of the sectional curvature, $D=A-\frac{1}{m}trA\cdot id$, then

$K\le\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA$, if $K\ge\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA$

$x(M)$ is Moebius equivalent to a Versonese surface $S^m(\sqrt{\frac{2(m+1)}{m}})$.

Then, we prove the theorem.

Proof of Theorem 1

Let $x:M^m\to S^n(n=m+p)$ is an submanifolds without umbilical point on unit sphere, K is the infimum of the sectional curvature, In Lemma 2, let $a=0$, then

$$0=\|\nabla B\|^2+\sum_{i,j,\alpha,t,k}B_{ij}^{\alpha}(B_{tk}^{\alpha}R_{tijk}+B_{ti}^{\alpha}R_{tkjk})-\sum_{\alpha,\beta}[tr(B_\alpha^2B_\beta^2)-tr(B_\alpha B_\beta)^2] \tag{12}$$

Because

$$\sum_{i,j,\alpha,t,k}B_{ij}^{\alpha}(B_{tk}^{\alpha}R_{tijk}+B_{ti}^{\alpha}R_{tkjk})\ge mK\|B\|^2,\quad \frac{\|B\|^4}{p}\le\sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2\le\|B\|^4 \tag{13}$$

then $0\ge\|\nabla B\|^2+mK\|B\|^2$

$$-\frac{1}{2}\{2\sum_{\alpha,\beta}[tr(B_\alpha^2B_\beta^2)-tr(B_\alpha B_\beta)^2]+\sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2\}+\frac{1}{2}\sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2$$

$$\ge\|\nabla B\|^2+mK\|B\|^2-\frac{1}{2}[1+\frac{1}{2}\operatorname{sgn}(p-1)]\|B\|^4+\frac{1}{2p}\|B\|^4 \tag{14}$$

Because $\|\nabla B\|^2\ge0$, Then

$$K-\frac{m-1}{2m^2}[1+\frac{1}{2}\text{sgn}(p-1)-\frac{1}{p}]\leq 0 \quad \textbf{(15)}$$

if $K\geq\frac{m-1}{2m^2}[1+\frac{1}{2}\text{sgn}(p-1)-\frac{1}{p}]$, then

$$K=\frac{m-1}{2m^2}[1+\frac{1}{2}\text{sgn}(p-1)-\frac{1}{p}] \quad \textbf{(16)}$$

Because $\|\nabla B\|=0$ $\phi=0$, according to the lemma 1, we get the following,

(i) $p=1$, $K=0$, $x(M)$ is Moebius equivalent to a Clifford minimal tori

$$S^k(\sqrt{\frac{k}{m}})\times S^{m-k}(\sqrt{\frac{m-k}{m}})\ ()1\leq k\leq m-1 \text{ in } S^{m+1},$$

(ii) $p=2$, $K=1/8$, $x(M)$ is Moebius equivalent to a Versonese surface in S^4

Proof of Theorem 2

Let $x:M^m\to S^n(n=m+p)$ is an submanifolds without umbilical point on unit sphere, K is the infimum of the sectional curvature, in Lemma 1, let $a=m/(m+2)$, from (13) and

$$|trDB^2|\leq\frac{m-2}{\sqrt{m(m-1)}}\|D\|$$

We obtain

$$\frac{\mathbf{2(m+1)m}}{\mathbf{m+2}}\mathbf{K}-\frac{\mathbf{m}}{\mathbf{m+2}}\cdot\frac{\mathbf{m(m-2)}}{\sqrt{\mathbf{m(m-1)}}}\|D\|-\frac{2m}{m+2}trA\leq 0 \quad \textbf{(17)}$$

Then

$$\mathbf{K}\leq\frac{\mathbf{m-2}}{\mathbf{2(m+1)}}\sqrt{\frac{\mathbf{m}}{\mathbf{m-1}}}\|\mathbf{D}\|+\frac{\mathbf{1}}{\mathbf{m+1}}\mathbf{trA} \quad \textbf{(18)}$$

If $K\geq\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA$

Then

$$K=\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA \quad \textbf{(19)}$$

From $\|\nabla B\|=0$ we obtain $\phi=0$,

Let $\{e_1,e_2,\cdots,e_m\}$ is a local standard orthogonal basis on TM, $A_{ij}=\lambda_i\delta_{ij}$, $D_{ij}=\overline{\lambda}_i\delta_{ij}$,

From (17) take the equal sign, get all the λ_i is equal to each other, let $\lambda_2 = ... = \lambda_m$, then $\tilde{\lambda}_2 = ... = \tilde{\lambda}_m$.

Then, we let

$$B_{11} = -(m-1)\mu, \ \ B_{22} = ... = B_{mm} = \mu,$$

$$\tilde{\lambda}_1 = \frac{m-1}{m}(\lambda_1 - \lambda), \ \tilde{\lambda}_2 = ... = \tilde{\lambda}_m = \frac{1}{m}(\lambda - \lambda_1)$$

$$\|D\| = \sqrt{\sum_i \tilde{\lambda}_i^2} = \frac{\sqrt{m(m-1)}}{m}|\lambda - \lambda_1| \quad \textbf{(20)}$$

from (6), we get

$$(m-1)^2\mu^2 + (m-1)\mu^2 = \frac{m-1}{m}, \ \mu = \pm\frac{1}{m} \quad \textbf{(21)}$$

when $\alpha \geq 2$, $B_{1\alpha} = 0$

because $B_{11} \neq B_{\alpha\alpha}$, we obtain

$$\omega_{\alpha 1} = 0$$

$$\sum_j B_{1\alpha,j}\omega_j = dB_{1\alpha} + \sum_j (B_{1j}\omega_{j\alpha} + B_{j\alpha}\omega_{j1}) = (B_{\alpha\alpha} - B_{11})\omega_{\alpha 1} = \mathbf{0} \quad \textbf{(22)}$$

$$-\frac{1}{2}\sum_{k,l} R_{1\alpha kl}\omega_k \wedge \omega_l = d\omega_{1\alpha} - \sum_k \omega_{1k} \wedge \omega_{k\alpha} = 0 \quad \textbf{(23)}$$

From (15)

$$0 = R_{1\alpha 1\alpha} = B_{11}B_{\alpha\alpha} + A_{11} + A_{\alpha\alpha} = -(m-1)\mu^2 + \lambda_1 + \lambda \quad \textbf{(24)}$$

$$\lambda_1 + \lambda = \frac{m-1}{m^2} \quad \textbf{(25)}$$

$$0 = \sum_j A_{1\alpha,j}\omega_j = dA_{1\alpha} + \sum_j A_{1j}\omega_{\alpha j} + \sum_j A_{\alpha j}\omega_{j1} = dA_{1\alpha} + (A_{11} - A_{\alpha\alpha})\omega_{1\alpha} \quad \textbf{(26)}$$

then $A_{1\alpha,j} = A_{1j,\alpha} = 0$ $A_{11,\alpha} = 0$, $A_{1\alpha,\alpha} = A_{\alpha\alpha,1} = 0$

$$A_{111}\omega_1 + \sum_\alpha A_{11,\alpha}\omega_\alpha = \sum_j A_{11,j}\omega_j \quad (2 \leq \alpha \leq m, 1 \leq j \leq m) \quad \textbf{(27)}$$

because of $0 = \sum_j \mathrm{A}_{11,j}\omega_j = dA_{11} + \sum_j A_{1j}\omega_{1j} + \sum_j A_{j1}\omega_{j1} = dA_{11}$, we get $\lambda_1 = const$.

If $\lambda \neq \lambda_1$ and $\omega_1 = \omega_2 = ... = \omega_m = 0$, assuming that $M^m = M_1^1 \times M_2^{m-1}$, K=0. From **(19)**, we obtain $K = \dfrac{m-1}{2m(m+1)}$, which is contradiction with K=0, then $\lambda_1 = \lambda, \tilde{\lambda}_1 = ... = \tilde{\lambda}_m = 0$, D=0.

Then we obtain R=K=$1/[m(m+1)]$. $x(M)$ is Moebius equivalent to minimal submanifold in S^n, then $\rho^2 = const$, $A_{ij} = \delta_{ij}/2$, From $g = \rho^2 dx \cdot dx$, we get $K = \rho^{-2} K_E$ and $\rho^{-2} = 2tr(A)/m$. From $trA = (1 + m^2 R)/2m$, we get $\rho^{-2} = (1/m^2) + R$, $K_E = \dfrac{m}{2(m+1)}$, Then we obtain $\overline{M}$ is isometric to the Veronese surface $S^m(\sqrt{2(m+1)/m})$.

ACKNOWLEDGMENT

This work is supported by the natural science foundation of Hebei Province (No.A2013209278), and the Scientific Technology Research

REFERENCES

[1] C.P.Wang; Moebius geometry of submanifolds in S^n [J], Manuscripta Math, **96**, 517-534 **(1998)**.

[2] H.Z.Li, C.P.Wang, F.E.Wu; A Moebius Characterization of Veronese surface in S^n [J], Math Ann, **319**, 707-714 **(2001)**.

[3] Z.Guo, H.Z.Li, C.P.Wang; The second variation formula for Willmore submanifolds in S^n Results Math, **40**, 205—225 **(2001)**.

[4] H.Z.Li, H.L.liu, C.P.Wang et al.; Moebius isoparametric hypersurfaces in S^{n+1} with two distinct principal curvatures[J], Acta Mathematica Sinica, English Series, **18(3)**, 437-446 **(2002)**.

[5] H.L.Liu, C.P.Wang, G.S.Zhao; Moebius isotropic submanifolds in S^n, Tohoku Math J, **53**, 553—569 **(2001)**.

[6] T.Itoh; Addendium to my paper On the Veronese manifolds, J.Math.Soc.Japan, **30**, 73-74 **(1978)**.

[7] H.U.Zejun, L.I.Haizhong; Submanifolds with constant Moebius scalar curvatue in S^n, Manuscripta Math, **111(3)**, 287-302 **(2003)**.