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Mathematical expression of 1D-nanodoping

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ABSTRACT

1D -nanodoping is supposed to be a perturbation generated by a sequence of delta Dirac pulses satisfying the relation $\pi\delta[\sin(\pi\xi)] = ?_n \delta(\xi - n)$ where n is an integer. Applications are discussed first for acoustic waves in a jerky flow, and for a scalar Bessel beam in a flow with a nanodoped velocity then for TE, TM fields inside a perfect conductor cylindrical wave gui-de with a nanodoped permittivity. We finally consider electromagnetic flashes.

KEYWORDS

Acoustic wave;
Subsonic flow;
Jerky velocity;
TE,TM fields;
Doped permittivity.

INTRODUCTION

The blossoming of nanotechnology during these last years[1] has generated a flow of ex-perimental and theoretical works in different domains of physics, chemistry, biology with often important new results. Of particular interest is the realization of nanodoped materials[2-4] as well as the analysis of slow light propagation in such structures[5-7]. And the behaviour of the electromagnetic fields E, H in a material doped with nano particles was previously analyzed[8].

We continue here this investigation for acoustic and electromagnetic wave propagation in a 2D material nanodoped in a direction, the doping being considered as realized by a sequence of perturbations made of delta Dirac pulses satisfying the relation[9]

$$\pi\delta[\sin(\pi\xi)] = \Sigma_n \delta(\xi - n) \quad (1.1)$$

n being an integer in $(-\infty.\infty)$. This distribution has a first derivative null

$$\partial\xi\delta[\sin(\pi\xi)] = 0 \quad (1.2)$$

With $y = \sin(\pi\xi)$ so that $dy = \pi(1-y^2)^{1/2} d\xi$ we get

$$\partial\xi\delta[\sin(\pi\xi)] = \pi(1-y^2)^{1/2}\delta'(y) \quad (1.3)$$

which becomes since $f(y)\,\delta'(y) = -f'(y)\,\delta(y)$

$$\partial\xi\delta[\sin(\pi\xi)] = \pi y(1-y^2)-^{1/2}\delta(y) \quad (1.4)$$

implying (1.2) using the relation $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$.

The applications of Eq.(1.1) are discussed first for acoustic waves in a jerky flow, and for a scalar Bessel beam in a flow with a nanodoped velocity then for TE, TM fields inside a per-fect conductor cylindrical wave guide with a nanodoped permittivity. We finally consider electromagnetic flashes. Each section is independent and can be read apart.

ACOUSTIC WAVES IN A 2D-SUBSONIC JERKY FLOW

Introduction

The description of mass flow requires two quantities: the density $\rho(x,t)$ and the velocity $v(x,t)$ from which the other properties of the flow are obtained. For a fluid in which viscosity and conductivity can be neglected, supposing in addition the flow isentropic so that the pression p is a function of ρ only, the sound waves satisfy the equations[10,11]

$$D\rho/Dt + \rho\nabla.v = 0, \quad D/Dt = \partial_t + v_j\partial_j, \ j = 1,2,3$$
$$Dv/dt + a^2/\rho\nabla\rho = 0, \quad a^2 = dp/d\rho \qquad (2.1)$$

1. Assuming first a fluid at rest with the density taking the value ρ_0 everywhere, we allow a disturbance to occur with a very small velocity and $\rho = \rho_0+\rho_1$ where ρ_1 is small. Then neglecting $\rho_1^{\,2}$ and $\rho_1 v$, the equations (2.1) reduce to the order $0(\rho_1^{\,2}, \rho_1 v)$ to

$$\partial_t\rho_1 + \rho_0\nabla.v = 0$$
$$\rho_0\partial_t v + a_0^{\,2}\,\rho_1 = 0, \quad a_0^{\,2} = (dp/d\rho)\rho_0 \qquad (2.2)$$

the speed of sound a_0 is the same everywhere in the flow.

Eliminating v from Eqs.(2.2) gives the wave equation

$$(\Delta - a_0^{-2}\partial_t^{\,2})\rho_1 = 0, \quad \Delta = \partial_x^{\,2} + \partial_y^{\,2} + \partial_z^{\,2} \qquad (2.3)$$

The disturbed density ρ_1 propagates as a wave with the velocity a_0.

2. Suppose now that the basis flow consists of steady velocity U parallel to the x-axis, then as-suming U, ρ_0 constant and the flow isentropic the equations (2.2) become[10] since v = U + u

$$\partial_t\rho_1 + U\partial_x\rho_1 + \rho_0\nabla.u = 0$$
$$\rho_0\partial_t U + \rho_0 U\partial_t u + a_0^{\,2}\,\nabla\rho_1 = 0 \qquad (2.4)$$

Eliminating u from Eqs.(2,4) gives the wave equation

$$(\Delta - a_0^{-2}\partial_t^{\,2})\rho_1 = a_0^{-2}\,(2U\partial^2_{x,t}\,\rho_1 + U^2\partial^2_x\,\rho_1) \qquad (2.5)$$

We are interested here in a 2D-flow so that Eq.(2,5) becomes

$$[(1 - a_0^{-2}U^2)\partial_x^{\,2} + \partial_z^{\,2} - a_0^{-2}\partial_t^{\,2}]\rho_1 = 2\,a_0^{-2}\,U\,\partial^2_{x,t}\,\rho_1 \qquad (2.6)$$

and we look for the solutions of this equation when U is the jerky velocity defined in the next section.

Jerky flow equation

The jerky velocity is defined by the periodic Dirac distribution in which U_0 is constant

$$U = U_0\,\delta[\sin(\pi\xi)]\ \xi = x/x_0 \qquad (2.7)$$

and, we suppose the flow subsonic

$$a_0^{-2}U_0^{\,2} << 1 \qquad (2.7a)$$

Now, Eqs.(2.5), (2.6) are valid for constant U, otherwise further terms requiring $\partial_x U$ would exist corresponding to ∇.U and (v.∇)U for arbitrary U[10]. But the first derivative of the jerky velocity is null according to (1.2)

$$\partial_x U = 0 \qquad (2.8)$$

which justifies the validity of Eq.(2.6) with the velocity (2.7).

Remark: Using the relation f(x) δ''(x–a) = f ''(a) δ(x–a) a similar calculation made to prove (1.2) gives easily $\partial\xi^2 U = -2\,\pi^2\,U$.

Then, taking into account (2.7) and (2.7a), the equation (2.6) becomes

$$[\partial_x^{\,2} + \partial_z^{\,2} - a_0^{-2}\partial_t^{\,2}]\rho_1 = 2\,a_0^{-2}\,U_0\,\delta[\sin(\pi\xi)]\,\partial^2_{x,t}\,\rho_1 \qquad (2.9)$$

The expression jerky flow comes from the series expansion (1.1) of the periodic Dirac distribution

$$U = U_0\,\Sigma_n\delta(\xi\pi - n\pi) \qquad (2.10)$$

n being an integer in (–∞,∞) so that (2.10) represents a sequence of nano-pulses leading the flow to move jerkily.

Jerky flow solutions

We look for the solutions of Eq.(2.9) in the form

$$\rho_1(x,z,t) = \exp(i\omega t + ik_z z)\,f(x) \qquad (2.11)$$

and substituting (2.11)into (2.9) gives

$$\{\partial_x^{\,2} + k_x^{\,2} - b\,\delta[\sin(\pi\xi)]\partial_x\}f(x) = 0 \qquad (2.12)$$

in which

$$k_x^{\,2} = a_0^{-2}\omega^2 - k_z^{\,2}, \qquad b = 2i\omega\,a_0^{-2}\,U_0 \qquad (2.12a)$$

And, since b is very small, we may write to the 0(b2) order

$$f(x) = f_1(x) + b\,f_2(x) + 0(b^2) \qquad (2.13)$$

Substituting (2.13) into (2.12) gives the two equations

$$(\partial_x^{\,2} + k_x^{\,2})f_1(x) = 0 \qquad a)$$
$$(\partial_x^{\,2} + k_x^{\,2})f_2(x) = \delta[\sin(\pi\xi)]\partial_x f_1(x) \qquad b) \qquad (2.14)$$

We get from (2.14a) with the amplitude A

$$f_1(x) = A\exp(ik_x x) \qquad (2.15)$$

and substituting (2.15) into (2.14b) gives

$$(\partial_x^{\,2} + k_x^{\,2})f_2(x) = ik_x A\,\delta[\sin(\pi\xi)]\,\exp(ik_x x) \qquad (2.16)$$

Then, using the expansion (2.10) of (2.7), the right hand side of Eq.(2.16) becomes

$$ik_x A\,\delta[\sin(\pi\xi)]\,\exp(ik_x x) = B\,\Sigma_n\,\delta(x - nx_0)\,\exp(ik_x x) \qquad (2.17)$$

with

$$B = i\pi k_x x_0^{-1}A \qquad (2.17a)$$

since

$$\delta(\xi\pi - n\pi) = \pi - 1x_0\,\delta(x - nx_0) \qquad (2.17b)$$

This suggests to look for the solutions of Eq.(2.16) in the form with constant amplitudes $f_{2,n}$

$$f_2(x) = \Sigma_n\,f_{2,n}\delta(x - nx_0) \qquad (2.18)$$

Substituting (2.18) into (2.16) and taking into account (2.17) give

$$(\partial_x^{\,2} + k_x^{\,2})f_{2,n}\,\delta(x - nx_0) = B\,(x - nx_0)\,\exp(ik_x nx_0) \qquad (2.19)$$

since

$$\delta(x-nx_0)\exp(ik_x x) = \delta(x-nx_0)\exp(ik_x nx_0) \qquad (2.19a)$$

But the relation $\phi(x)\,\delta''(x-a) = f''(a)\,\delta(x-a)$ implies

$$f_{2,n}\partial_x^2\delta(x-nx_0) = 0 \qquad (2.19b)$$

so that we get at once from (2.19)

$$f_{2,n} = k_x^{-2}B\exp(ik_x nx_0) \qquad (2.20)$$

and (2.18) becomes taking into account (2.17a) and (2.17b)

$$f_2(x) = \Sigma_n k_x^{-2}B\exp(ik_x nx_0)\,\delta(x-nx_0) = i\,k_x^{-1}A\,\Sigma_n\exp(ik_x x)\,\delta(\xi\pi - n\pi) \qquad (2.21)$$

and finally according to (2.10)

$$f_2(x) = i\,k_x^{-1}A\exp(ik_x x)\,\delta[\sin(\pi\xi)] \qquad (2.22)$$

Then substituting (2.15) and (2.22) into (2.13) gives the $0(b^2)$ approximation

$$f(x) = A\exp(ik_x x)\{1 + ib\,k_x^{-1}\delta[\sin(\pi\xi)]\} \qquad (2.23)$$

and according to (2.11) and (2.23) with $k_x^2 + k_z^2 = a_0 - 2\omega^2$

$$\rho_1(x,z,t) = A\exp(i\omega t + ik_z z + ik_x x)\{1 + ib\,k_x^{-1}\delta[\sin(\pi\xi)]\} \qquad (2.24)$$

So, in a jerky flow, the disturbed density ρ_1 appears as an harmonic plane wave nanodoped by Dirac pulses at regular intervalls.

Discussion

The term jerky flow is generally attached to the Portevin-Le Chatelier effect, discovered in 1923, and corresponding to a kind of plastic instability observed in many dilute alloys at cer-tain ranges of strain and temperature[12] and, very often escorted by acoustic emissions[13].

The jerky flow introduced in this work is at half way between a steady flow and a chaotic flow : on one hand, this flow is not steady state since the relations $\partial_x U = 0$, $\partial^2_x U \neq 0$, proved in Sec.2, can be easily gen-eralized to $\partial_x^{2p+1}U = 0$, $\partial_x^{2p}U \neq 0$, p = 0,1,2… On the other hand, it may hardly be considered as chaotic since its disturbances, due to the periodic Dirac distri-bution, regularly scattered along ox, have an exceedingly thin thickness.. And, in this situa-tion, the amplitudes of sound waves, described by harmonic plane waves, suffer ac-cording to (2.23), from the same staccatto disturbances as the jerky flow.

We may imagine a jerky flow as water flowing down a starcase with a low velocity to avoid turbu-lences[14].

BESSEL SCALAR BEAM IN A FLOW WITH A NANODOPED VELOCITY

Introduction

We are interested in a Bessel scalar beam $\psi(r,z,t)$, cylindrical around oz, solution of the wa-ve equation

$$[(\partial_r + 1/r)\partial_r + \partial_z^2 - n^2(z)\partial_t^2]\psi(r,z,t) = 0 \qquad (3.1)$$

in which $n-1(z)$ is a velocity function of z, and $\psi(r,z,t)$ an acoustic field. In particular for $n = n_0$ = constant, this equation has the solution with the parameters k, α, β

$$\psi(r,z,t) = J_0(kr)\exp[-in_0(\alpha-\beta)z]\exp[i(\alpha+\beta)t] \qquad (3.2)$$

J_0 is the Bessel function of the first kind of order zero and

$$k^2 = 4n_0^2\alpha\beta \qquad (3.2a)$$

Then, when n is not constant, we look for the solution of Eq.(3.1) in the form

$$\psi(r,z,t) = J_0(kr)\exp[i(\alpha+\beta)t]\,\phi(z) \qquad (3.3)$$

and, substituting (3.3) int (3.1) gives the differential equa-tion satisfied by $\phi(z)$

$$\phi''(z) + [(\alpha+\beta)^2n^2(z) - k^2]\,\phi(z) = 0 \qquad (3.4)$$

Suppose now that $n^2(z)$, depending on two constants n_0^2, $n_1^2 << n_0^2$, has the $0(n_1^4)$ approxi-mation in which f(z) is a bounded function

$$n^2(z) = n_0^2 + n_1^2 f(z) + 0(n^4) \qquad (3.5)$$

so that the equation (3.4) becomes

$$\phi''(z) + [\gamma_0^2 + \gamma_1^2(z)]\,\phi(z) = 0 \qquad (3.6)$$

with, taking into account (3.2a)

$$\gamma_0^2 = n_0^2(\alpha+\beta)^2 - k^2 = n_0^2(\alpha-\beta)^2,\quad \gamma_1^2(z) = (\alpha+\beta)^2 f(z) \qquad (3.6a)$$

This result suggests to look for the solutions of Eq.(3.6) in the form

$$\phi(z) = \phi_0(z) + n_1^2\phi^1(z) + 0(n_1^4) \qquad (3.7)$$

and substituting (3.7) into (3.6) gives the two equations

$$\phi_0''(z) + \gamma_0^2\phi_0(z) = 0 \qquad (3.8a)$$

$$\phi_1''(z) + \gamma_0^2\phi_1(z) = -\gamma_1^2(z)\,\phi_0(z) \qquad (3.8b)$$

The equation (3.8a) has the solutions in which A, B are arbitrary amplitudes

$$\phi_0(z) = A\exp(i\gamma_0 z) + B\exp(-i\gamma_0 z) \qquad (3.9)$$

So, we are left with the nonhomogeneous equation (3.8b) to be solved when f(z) is a sequence of delta Dirac pulses. And, to make calculations easier to check, we give, in terms of length L and time T, a dimensional analysis of the main parameters, using the conventional

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notation [E] for the dimension of E.

$[n] = [n_0] = [n_1] = L{-}1T; [\alpha] = [\beta] = T{-}1;$
$[k] = L{-}1; [\gamma_0] = L{-}1, [\gamma_1] = T{-}1 ; [f] = L^0T^0$ (3.10)

Bessel beam in a flow with nanodoped velocity

To solve Eq.(3.8b), we look for its solutions in the form[15]

$$\phi_1(z) = c_1(z)\, w_1(z) + c_2(z)\, w_2(z) \quad (3.11)$$

where $w_1(z) = \exp(i\gamma_0 z)$, $w_2(z) = \exp(-i\gamma_0 z)$ are solutions of the homogeneous equation

$$\phi_1''(z) + {}_0^2\, \phi_1(z) = 0 \quad (3.11a)$$

Then[15] taking into account the right hand side of (3.8b) the derivatives $c'_{1,2}(z)$ of $c_{1,2}(z)$ are

$$c'_1(z) = \gamma_1^2(z)\, \phi_0(z)\, w_2(z)\, [w_1(z)\, w'_2(z) - w'_1(z)\, w_2(z)]^{-1}$$
$$= (i\gamma_1^2(z)/2\gamma_0)\, {}_0(z) \exp(-i\gamma_0 z) \quad (3.12a)$$

and similarly

$$c'_2(z) = -\gamma_1^2(z)\, \phi_0(z)\, w_1(z)\, [w_1(z)\, w'_2(z) - w'_1(z)\, w_2(z)]^{-1}$$
$$= -(i\gamma_1^2(z)/2\gamma_0)\, \phi_0(z) \exp(i\gamma_0 z) \quad (3.12b)$$

Substituting (3.9) into (3.12a,b) gives

$$c'_1(z) = (i\gamma_1^2(z)/2\gamma_0)\, [A + B\exp(-2i\gamma_0 z)]$$
$$c'_2(z) = -(i\gamma_1^2(z)/2\gamma_0)\, [A\exp(2i\gamma_0 z) + B] \quad (3.13)$$

We now suppose that $f(z)$ is a sequence of delta pulses $\Sigma\nu\, \delta(z/z_0-\nu)$ where ν is an integer in $(-\infty, \infty)$, δ the Dirac distribution and z_0 a constant.

Then, according to (3.6a) and in agreement with (3.10)

$$\gamma_1^2(z) = (\alpha+\beta)^2\, \Sigma_\nu\, \delta(z/z_0-\nu) \quad (3.14)$$

and writing

$$c'_1(z) = \Sigma_\nu\, c'_{1,}\nu(z), \qquad c'_2(z) = \Sigma_\nu\, c'_{2,}\nu(z) \quad (3.15)$$

we get from (3.13) and (3.14)

$$c'_{1,\nu}(z) = i[(\alpha+\beta)^2/2\gamma_0]\, \delta(z/z_0-\nu)$$
$$[A + B\exp(-2i\gamma_0 z)] \quad (3.16a)$$

and, using the properties of the Dirac distribution

$$c'_{1,\nu}(z) = iz_0[(\alpha+\beta)^2/2\gamma_0]$$
$$[A + B\exp(-2i\gamma_0 \nu z_0)]\, \delta(z-\nu z_0) \quad (3.16b)$$

and similarly

$$c'_{1,\nu}(z) = i[(\alpha+\beta)^2/2\gamma_0]\, \delta(z/z_0-\nu)$$
$$[A\exp(2i\gamma_0 z) + B] \quad (3.17a)$$

$$c'_{1,\nu}(z) = iz_0[(\alpha+\beta)^2/2\gamma_0]$$
$$[A\exp(2i\gamma_0 \nu z_0)+B]\, \delta(z-\nu z_0) \quad (3.17b)$$

so that

$$c_1(z) = \Sigma_\nu\, c_{1,\nu}(z), \qquad c_2(z) = \Sigma_\nu\, c_{2,\nu}(z) \quad (3.18)$$

with according to (3.16b), (3.17b)

$$c_{1,\nu}(z) = iz_0[(\alpha+\beta)^2/2\gamma_0]$$
$$[A + B\exp(-2i\gamma_0 \nu z_0)]\, U(z-\nu z_0)$$

$$c_{1,\nu}(z) = -iz_0[(\alpha+\beta)^2/2\gamma_0]$$
$$[A\exp(2i\gamma_0 \nu z_0)+B]\, U(z-\nu z_0) \quad (3.19)$$

in which $U(z-\nu z_0)$ is the unit step function.

Substituting (3.19) into (3.18) and taking (3.11) into account achieves to determine $\phi_1(z)$.

These caculations are formal since nothing is known on the convergence of the series (3.18) and this unsatisfactory situation comes from the definition (3.14) of $\gamma_1^2(z)$ by an infinite se-ries. But, according to the relation (1.1) rewritten below for convenience

$$\pi\delta[\sin(\pi z/z_0)] = \Sigma_n\, \delta(z/z_0-n) \quad (3.20)$$

we get

$$\gamma_1^2(z) = \pi\, (\alpha+\beta)^2\, \delta[\sin(\pi z/z_0)] \quad (3.21)$$

Then, using (3.16a), (3.17a) instead (3.16b), (3.17b), the function $c'_{1,2}(z)$ are no more defined by the series (3.15) but become

$$c'_1(z) = i\pi[(\alpha+\beta)^2/2\gamma_0]\, \delta[\sin(\pi z/z_0)]$$
$$[A + B\exp(-2i\gamma_0 z)] \quad (3.22a)$$

$$c'_2(z) = i\pi[(\alpha+\beta)^2/2\gamma_0]\, \delta[\sin(\pi z/z_0)]$$
$$[A\exp(2i\gamma_0 z) + B] \quad (3.22b)$$

The price to pay is to perform the integration of (22a,b). Let us for instance consider the first term of (3.22a)

$$[c'_1(z)]_1 = i\pi A[(\alpha+\beta)^2/2\gamma_0]\, \delta[\sin(\pi z/z_0)] \quad (3.23)$$

with the integration

$$[c_1(z)]_1 = i\pi A[(\alpha+\beta)^2/2\gamma_0] \int_0^z \delta[\sin(\pi s/z_0)]\, ds \quad (3.24)$$

Let us introduce the functions

$$u = [\sin(\pi s/z_0), \qquad v = [\sin(\pi z/z_0) \quad (3.25)$$

so that

$$du = \pi/z_0\, (1-u^2)^{1/2}\, ds \text{ a)},\ \partial_z = \pi/z_0\, (1-v^2)^{1/2}\partial_v \text{ b)} \quad (3.26)$$

Then, using (3.26a) the integral (3.24) becomes

$$[c_1(z)]_1 = i\pi A z_0/v\, [(\alpha+\beta)^2/2\gamma_0]\, I_1(v) \quad (3.27)$$

$$I_1(v) = \int_0^v \delta(u)\, (1-u^2)^{-1/2}\, du \quad (3.27a)$$

taking into acccount (3.26b) we get at once $\partial_z[c_1(z)]_1 = [c'_1(z)]_1$ while according to (3.27a)

$$I_1(v) = (1-u^2)^{-1/2}\, U(v) + \int_0^v U(u)\, (1-u^2)^{-1/2}\, u\, du \quad (3.28)$$

the integral in (3.28) needs a numerical approximation. The integration of the second term of (3.22a)

$$[c'_1(z)]_2 = i\pi B[(\alpha+\beta)^2/2\gamma_0]$$
$$\delta[\sin(\pi z/z_0)] \exp(-i\gamma_0 z) \quad (3.29)$$

gives

$$[c_1(z)]_2 = i\pi B[(\alpha+\beta)^2/2\gamma_0]$$
$$\int_0^z \delta[\sin(\pi s/z_0)] \exp(-i\gamma_0 s)\, ds \quad (3.30)$$

and, with (3.25), (3.26a) this integral becomes

$$[c_1(z)]_2 = i\pi Bz_0/v\,[(\alpha+\beta)^2/2\gamma_0]\,I_2(v) \quad (3.31)$$

$$I_2(v) = \int_0^v \delta(u)\,\Phi(u)\,du,$$
$$\Phi(u) = (1-u^2)^{-1/2}\exp[(-i\gamma_0 z_0/\pi)\,\arcsin u] \quad (3.31a)$$

and

$$I_2(v) = U(v)\,\Phi(v) - \int_0^v U(u)\,\Phi'(u)\,du \quad (3.32)$$

The integral in (3.32) requires also a numerical approximation. We have, of course a similar result for the expression (3.27b) of $c_2'(z)$.

Discussion

To manage the scalar wave equation with a variable velocity v(z), we had to assume that $n^2(z)$ [= v–2(z)] has the approximation $0(n_1^4)$ and also to generalize this approximation to the solutions of Eq.(3.4) supposing in particular f(z) bounded.The technique used to solve the inhomogeneous differential equation (3.8b) is general, the only difficulties may come from the integration of the coefficients $c'_{1,2}(z)$, a numerical integration could be needed (which is not the case if f(z) is a trigonometric function).

The Bessel solution (3.2) is the Durnin scalar nondiffracting beam[16].

TE,TM FIELDS IN A NANODOPED WAVE GUIDE

Maxwell equations

We consider a perfect conductor, cylindrical waveguide of z-axis and radius a, endowed with the nanodoped permittivity in which m is an integer in $(-\infty,\infty)$

$$\varepsilon(z) = \varepsilon_0 + \eta\, z_0 \Sigma_m\, \delta(z - mz_0), \qquad 0 = r = a \quad (4.1)$$

ε_0, η are constant, δ the Dirac distribution and according to (1.1) and (1.2)

$$\partial_z \varepsilon(z) = 0 \quad (4.1a)$$

We work with the cylindrical coordinates r, θ, z. The fields do not depend on θ and the permeability μ is constant.

Then, the Maxwell curl equations are[17]

$$\begin{aligned}
&-\partial_z E\theta + \mu\partial_t H_r = 0 && a)\\
&(\partial_r + 1/r)\, E\theta + \mu\partial_t H_z = 0 && b)\\
&\partial_z E_r - \partial_r E_z + \mu\partial_t H\theta = 0 && c) \quad (4.2)
\end{aligned}$$

$$\begin{aligned}
&\partial_z H\theta + \varepsilon(z)\,\partial_t E_r = 0 && a)\\
&(\partial_r + 1/r)\, H\theta - \varepsilon(z)\partial_t E_z = 0 && b)\\
&\partial_z H_r - \partial_r H_z - \varepsilon(z)\partial_t E\theta = 0 && c) \quad (4.3)
\end{aligned}$$

These equations divide into two sets : TE (H_r, H_z, Eθ) and TM (E_r, E_z, Hθ) fields.

From a dimensional analysis viewpoint, which is interesting to foresee some results and to avoid some bugs, we have, using the conventional notation [F] for the dimension of F.

$$[E] = ML^2T^{-2}Q^{-1},\ [D] = L^{-1}Q,$$
$$[\varepsilon] = [\eta] = M^{-1}L^{-3}T^2Q^2,\ [H] = T^{-1}Q,$$
$$[B] = MLT^{-1}Q^{-2},\ [\mu] = MLQ^{-2},\ [\delta(z)] = L^{-1} \quad (4.4)$$

in which L, M, T, Q denote length, mass, time and charge. These results come from the Strat-ton dimensional analysis[18] by changing Q into LQ in the dimensions of the field densities since one has here to deal with a 2D-space.

Now, substituting (4.2a,b)}into the time derivative of (4.3c) gives the wave equation fulfilled by Eθ

$$[\partial_z^2 + \partial_r(\partial_r + 1/r) - n^2(z)\,\partial_t^2]\, E\theta = 0 \quad (4.5)$$

Substituting similarly (4.3a,b) into (4.2c) and using (4.1a) shows that Hθ satisfies also the wave Eq.(4.5).

So, we are left with this equation to be solved. Let us remark that for a scalar field, the term $\partial_r(\partial_r+1/r)$ in (4.5) is changed into $(\partial_r+1/r)\partial_r$ at the origin of some confusion[19].

Boundary conditions

We look for the solutions of Eq.(4.5) in the following form with arbitrary amplitudes A_e, A_h

$$\{E_\theta, H_\theta\} = \{A_e, A_\theta\}\,\psi(r,z) \quad (4.6)$$

$$\psi(r,z) = J_1(kr)\,\phi(z)\exp(i\omega t) \quad (4.6a)$$

J_1 is the Bessel function of the first kind of order one and φ(z) a function to be determined. From a dimensional viewpoint :

$$[A_e] = MLT^{-2}Q^{-1}, \quad [A_h] = L^{-1}T^{-1}Q, \quad [\psi] = L \quad (4.6b)$$

Now, the solutions must satisfy the boundary conditions on the perfect conductor surface r = a

$$TE: [\partial_r H_z(r,z)]_{r=a} = 0, \qquad TM: [E_z(r,z)]_{r=a} = 0 \quad (4.7)$$

But, according to (4.2b), (4.3b)

$$\begin{aligned}
&TE: i\omega\mu H_z = -(\partial_r + 1/r)E\theta && a),\\
&TM: i\omega\varepsilon(z)E_z = (\partial_r + 1/r)H\theta && b) \quad (4.8)
\end{aligned}$$

Then, substituting (4.6a) into (4.8a,b), performing the ∂_r-derivative of (4.8a) and using the relations

$$\begin{aligned}
&(\partial_r + 1/r)\, J_1(kr) = k\, J_0(kr) && a)\\
&\partial_r(\partial_r + 1/r)\, J_1(kr) = -\, k^2\, J_1(kr) && b) \quad (4.9)
\end{aligned}$$

one checks at once that the boundary conditions (4.7) are fulfilled if

$$TE: ka = p_l, \qquad TM: ka = q_l \quad (4.10)$$

in which p_l, q_l are the lth roots of $J_1(x) = 0$ and $J_0(x) = 0$ respectively.

Finally, substituting (4.6a) into the wave equation (4.5) and using (4.9b) give the differential equation satisfied by $\phi(z)$ where $n^2(z) = \mu\varepsilon(z)$

$$\phi'' - [k^2 - \omega^2 n^2(z)]\,\phi = 0 \qquad (4.11)$$

which is a Mathieu-like equation[20] that we have now to solve.

Mathieu equation

Let us start with the simple refractive index

$$n^2(z) = \mu\varepsilon_0 + \mu\eta z_0 \delta(z-mz_0) \qquad (4.12)$$

transforming Eq.(4.11) into

$$\phi'' - [\lambda^2 - \alpha^2 \delta(z-mz_0)]\phi = 0 \qquad (4.13)$$

with

$$\alpha^2 = \omega^2 \mu\eta z_0 \quad a), \quad \lambda^2 = k^2 - \omega^2\mu\varepsilon_0 \quad b) \qquad (4.14)$$

We look for the solution of (4.13 in the form

$$\phi_m(z) = \exp[\lambda(z-mz_0)]\,U(z-mz_0) \qquad (4.15)$$

in which U(z) is the unit step function with U(0) = 1 so that $\phi_m(mz_0) = 1$ and

$$\delta(z-mz_0)\,\phi_m(z) = \delta(z-mz_0) \qquad (4.15a)$$

Now, a simple calculation gives

$$\begin{aligned}\phi_m'' - \lambda^2\phi_m &= \lambda \exp[\lambda(z-mz_0)]\,\delta(z-mz_0)\\ &= \lambda\,\delta(z-mz_0) \qquad (4.16)\end{aligned}$$

Taking into account (4.14a,b) the comparison of (4.13) and (4.16) gives

$$\lambda = -\alpha^2 = -\omega^2\mu\eta z_0 \qquad (4.17)$$

supplying the equation

$$\omega^4\mu^2\eta^2 z_0^2 + \omega^2\mu\varepsilon_0 - k^2 = 0 \qquad (4.18)$$

with the solution

$$\omega^2 = [-\varepsilon_0 + (\varepsilon_0^2 + 4k^2\eta^2 z_0^2)^{1/2}]\,(2\mu\eta^2 z_0^2)^{-1} \qquad (4.19)$$

in which according to (4.10) $k = p_l/a$ or $k = q_l/a$; l = 1,2 3…

We now consider the permittivity

$$\varepsilon(z) = \varepsilon_0 + \eta\, z_0 \Sigma_m \delta(z - mz_0),\ m = 0,1,2\ldots \qquad (4.20)$$

which is (4.1) limited to the positive integers. Then, the equation (4.11) becomes with α^2, λ^2 given by (4.14)

$$\phi'' - [\lambda^2 - \alpha^2 \Sigma_m \delta(z-mz_0)]\,\phi = 0 \qquad (4.21)$$

Let us look for the solutions of (4.21) in the form

$$\begin{aligned}&\phi(z) = \Sigma_j \phi_j(z)\ j = 0,1,2\ldots,\\ &\phi_j(z) = \exp[\lambda_j(z-jz_0)]\,U(z-jz_0) \qquad (4.22)\end{aligned}$$

Substituting (4.22) into (4.21) gives with j = 0,1,2…, m = 0,1,2…

$$\Sigma_j [\phi_j''(z) - \lambda_j^2 \phi_j(z) + \alpha_j^2 \Sigma_m (z-mz_0)\,\phi_j(z)] = 0 \qquad (4.23)$$

in which

$$\alpha_j^2 = \omega_j^2 \mu\eta z_0 \quad a), \qquad \lambda_j^2 = k^2 - \omega_j^2\mu\varepsilon_0 \quad b) \qquad (4.23a)$$

with according to (4.16)

$$\begin{aligned}\phi_j''(z)\,\lambda_j^2\phi_j(z) &= \lambda_j \exp[l_j(z-jz_0)]\,\delta(z-jz_0)\\ &= \lambda_j (z-jz_0) \qquad (4.24)\end{aligned}$$

Taking into account (4.24), Eq.(4.23) becomes

$$\Sigma_j \lambda_j \delta(z-jz_0) + \Sigma_j \Sigma_m \alpha_j^2 \delta(z-mz_0)\,\phi_j(z) = 0 \qquad (4.25)$$

Exchanging j and m in the second term on the left hand side of (4.25) gives

$$\Sigma_j \delta(z-jz_0)\,[\lambda_j + \Sigma_m \alpha_m^2 \phi_m(jz_0)] = 0 \qquad (4.26)$$

implying

$$\lambda_j + \Sigma_m \alpha_m^2 \phi_m(jz_0)] = 0 \qquad (4.27)$$

in which acccording to (4.22)

$$\phi_m(jz_0)] = \exp[\lambda_m(j-m)z_0]\,U[(j-m)z_0] \qquad (4.27a)$$

This expression is null for m > j and $\phi_m(jz_0) = \exp[\lambda_m(j-m)z_0]$ for m = j. Using this result and making m = j, j–1, j–2… the relation (4.27) becomes

$$\begin{aligned}&\lambda_j + \alpha_j^2 + \alpha_j{-}1^2 \exp(\lambda_j{-}1\ z_0) +\\ &\alpha_{j-2}^{\ 2} \exp(\lambda_{j-2}\ z_0) + \ldots = 0 \qquad (4.28)\end{aligned}$$

giving $\lambda_0 + \alpha_0^2 = 0$, $\lambda_1 + \alpha_1^2 + \alpha_0^2 \exp(\lambda_0 z_0) = 0$, …

These relations determine the frequency bands ω_j in which the solutions (4.22) exist.

Then, substituting (4.22) into (4.6a) gives the components Eθ, Hθ, of the TE and TM fields

$$\begin{aligned}&E\theta\,(r,z,t) = \Sigma_j A_{e,j} J_1(kr)\,\phi_j(z) \exp(i\omega_j t) \quad a)\\ &H\theta\,(r,z,t) = \Sigma_j A_{h,j} J_1(kr)\,\phi_j(z) \exp(i\omega_j t) \quad b) \qquad (4.29)\end{aligned}$$

We have still to get $\{H_r, H_z\}$ and $\{E_r, E_z\}$. Substituting (4.29a) into (4.2a,b) gives the compo-nents $\{H_r, H_z\}$ of the TE field

$$\begin{aligned}&i\omega_l\mu\, H_r = \Sigma_j A_{e,j} J_1(kr)\,\partial_z\phi_j(z) \exp(i\omega_j t)\\ &i\omega_l\mu\, H_z = -k\,\Sigma_j A_{e,j} J_0(kr)\,\phi_j(z) \exp(i\omega_j t) \qquad (4.30)\end{aligned}$$

Similarly, substituting ((4.29b) into (4.3a,b), we get

$$\begin{aligned}&i\omega_l\varepsilon(z)\, H_r = -\Sigma_j A_{h,j} J_1(kr)\,\partial_z\phi_j(z) \exp(i\omega_j t)\\ &i\omega_l\varepsilon(z)\, H_z = \Sigma_j A_{h,j} J_0(kr)\,\phi_j(z) \exp(i\omega_j t) \qquad (4.31)\end{aligned}$$

Discussion

The wave guide analyzed in this work has a particular structure: first, the inner permittivity for 0 = r = a is nanodoped according to the relation (4.20) which is (4.1) limited to the posi-tive integers: the general case has still to be solved. Second, its interior is partitionned by a set of virtual disks with radius a, dielectric constant η and exceedingly thin thickness simulating delta distributions, regularly distributed along the z axis. This brief description of the inner wave guide suggests that such a structure could be experimentally realized at least approximately.

One may intuitively imagine that an electromagnetic wave propagating inside the partitio-ned wave guide will vibrate along oz acccording to the partitioning periodicity. This would explain why $\phi(z)$ satisfies a Mathieu equation, introduced by Emile Mathieu, a long time ago, to analyze the vibrations of elliptic membranes. ït is known[21] that the standard Mathieu equation{similar to (4.21) with $\delta(z-z_0)$ changed into $2\cos[b(z-z_0)]$}has an infinity of periodic solutions,

Then, suppose that one could realize a nanodoped wave guide, as described above, of length L between z = 0 and z = L, it should be possible to check the behaviour of TE, TM fields obtained by illuminating the z = 0 face with an harmonic plane wave, its electric field being parallel or perpendicular to the z axis and to compare experiment and theory.

ELECTROMAGNETIC NANO FLASHES

Introduction

Changing ξ into $\tau = t/t_0$, Eq.(1.1) becomes

$$\pi\delta[\sin(\pi\tau)] = \Sigma_n \delta(\tau - n) \quad (5.1)$$

describing a sequence of Dirac pulses in time which are identified with light flashes.

Now, in a homogeneous isotropic medum with $B = \mu H$, $D = \varepsilon E$, the Maxwell equations with cylindrical co-ordinates, r, θ, z are, for fields not depending on θ

$$\begin{aligned} &-\partial_z E\theta + \mu\partial_t H_r = 0 && a)\\ &(\partial_r + 1/r)\, E\theta + \mu\partial_t H_z = 0 && b)\\ &\partial_z E_r - \partial_r E_z + \mu\partial_t H\theta = 0 && c)\\ &(\partial_r + 1/r)\, H_r + \partial_z H_z = 0 && d) \quad (5.2)\end{aligned}$$

$$\begin{aligned} &\partial_z H\theta + \varepsilon\partial_t E_r = 0 && a)\\ &(\partial_r + 1/r)\, H\theta - \varepsilon\partial_t E_z = 0 && b)\\ &\partial_z H_r - \partial_r H_z\ \varepsilon\partial_t E\theta = 0 && c)\\ &(\partial_r + 1/r)\, E_r + \partial_z E_z = 0 && d) \quad (5.3)\end{aligned}$$

These equations divide into two sets: TE (H_r, H_z, $E\theta$) and TM (E_r, E_z, $H\theta$) fields. We are inte-rested into the flashing TE, TM plane waves with the flash functions (5.1)

TE, TM flashes

We have according to (1.1)

$$d/d\tau\, \delta[\sin(\pi\tau)] = 0 \quad (5.4)$$

Then, we look for the solutions of Eq.(5.2), (5.3) in the form

$$\{E(r, z,t), H(r, z, t)\} = \{E^*(r, z), H^*(r, z)\}\, \delta[\sin(\pi\tau)] \quad (5.5)$$

so that according to (5.4)

$$\partial_t\{E(r, z,t), H(r, z, t)\} = 0 \quad (5.6)$$

Substituting (5.5) into (5.2a,b) and (5.3c), taking into account (5.6) give

$$E\theta^*(r, z) = 0,\ \partial_z H_r^*(r, z) - \partial_r H_z^*(r, z) \quad (5.7)$$

with the solution satisfying the divergence equation (5.2d) :

$$\begin{aligned} &H_r^*(r, z) = \underline{H} \exp(ikz)\, J_1(kr),\\ &H_z^*(r, z) = -i\, \underline{H} \exp(ikz)\, J_0(kr) \quad (5.8)\end{aligned}$$

$\underline{H}$ is an arbitrary amplitude and k an arbitrary wave vector.

We have a similar result from (5.3a,b,d) and (5.2c) for $H\theta$, E_r, E_z

$$\begin{aligned} &H\theta^*(r, z) = 0,\ E_r^*(r, z) = \underline{E} \exp(ikz)\, J_1(kr),\\ &E_z^*(r, z) = -i\, \underline{E} \exp(ikz)\, J_0(kr) \quad (5.9)\end{aligned}$$

so that taking into account (5.6), (5.7), (5.9) and $\tau = t/t_0$, we finally get

$$\begin{aligned} &E\theta(r, z, t) = H\theta(r, z, t) = 0\\ &\{E_r(r, z,t), H_r(r, z, t)\} = \{\underline{E}, \underline{H}\}\exp(ikz)\, J_1(kr)\, \delta[\sin(\pi t/t_0)]\\ &\{E_z(r, z,t), H_z(r, z, t)\} = -i\, \{\underline{E}, \underline{H}\}\exp(ikz)\, J_0(kr)\, \delta[\sin(\pi t/t_0)] \quad (5.10)\end{aligned}$$

k has not the same value in TE and TM modes.

These solutions represent plane waves flashing at regular times nt_0.

Discussion

The TE and TM flashing waves has a characteristic feature : the θ-component of the elec-tric (magnetic) field is null for TE (TM) modes. So, reduced to the r, z components of electric and magnetic fields, they are fully confined in the r, z-plane. As a consequence, no relation exists between the circular frequency ω and the wave vector k which may be obtained from boundary conditions. Suppose for instance, that these flashing waves propagate inside a perfect conductor cylindrical waveguide of z-axis and radius a. The boundary conditions on the electromagnetic field are

$$TE : [\partial_r H_z]_{r=a} = 0, \qquad TM : [E_z]_{r=a} = 0 \quad (5.11)$$

that is according to (5.8), (5.9)

Full Paper

$$\text{TE}: J_1(ka) = 0, \qquad \text{TM}: J_0(ka) = 0 \qquad (5.12)$$

implying

$$\text{TE}: ka = p_l, \qquad \text{TM}: ka = q_l \qquad (5.13)$$

in which p_l, q_l are the l th roots of $J_1(x) = 0$ and $J_0(x) = 0$.

The periodic Dirac distribution (5.1) has a simple frequency representation $\Phi(\omega)$

$$\Phi(\omega) = 1/t_0 \int_{-\infty}^{\infty} \exp(i\omega t)\, \delta[\sin(\pi t/t_0)]\, dt \qquad (5.14)$$

Introducing the variable $T = \sin(\pi t/t_0)$ such as

$$dT = \pi/t\,(1-T^2)^{1/2}\, dt,\ t = (-1^n)t_0/\pi \arcsin T + nt_0 \qquad (5.15)$$

in which n is an integer in $(-\infty,\infty)$ the Fourier transform (5.14) becomes

$$\Phi(\omega) = \Sigma_n\, \Phi_n(\omega) \qquad (5.16)$$

$$\Phi_n(\omega) = 1/t_0 \int_{-\infty}^{\infty} \exp\{i\omega [(-1^n)t_0/\pi \arcsin T + nt_0]\, \delta(T)\, dT \qquad (5.16a)$$

that is

$$\Phi_n(\omega) = 1/\pi \exp(i\omega n t_0) \qquad (5.17)$$

and, substituting (5.17) into (5.16) gives

$$\Phi(\omega) = \Sigma_n 1/\pi \exp(i\omega n t_0) = 1/\pi\,(1 - \exp(i\omega t_0))^{-1} \qquad (5.18)$$

On the other hand, substituting the series (5.1) into (5.14) give

$$\Phi(\omega) = 1/\pi \int_{-\infty}^{\infty} \Sigma_n \exp(i\omega t)\, \delta(t-nt_0)\, dt \qquad (5.19)$$

in agreement with (5.18) if the series in (5,19) can be integrated term by term, which justifies, a posteriori, the exchange of integration and summation.

CONCLUSIONS

That 1D-nanodoping may be described by a sequence of delta Dirac pulses is rather natural. In a doped material, one of its properties, mechanical, chemical, electric, magnetic… is per-turbed by the adjunction of a substance to improve the performances of this property. Nano-doping intervenes when the dimension of this substance is much more smaller than the dimension of the medium in which this property has to be improved. In 1D-nanodoping, the injected substance may be considered as a dimensionless dot, correctly represented by a Dirac distri-bution.

Once accepted this representation of 1D-nanodoping, a mathematical simulation of this situation, based on the conventional equations of mechanics, acoustics, electromagnetism has to be performed to forsee the consequences of this nanodoping. To estimate the performances of 1D-nanomaterials, it is first necessary to get manageable analytical solutions of this simulaion

As shown here, this result is reached at the expense of sometimes rather severe approxima-tions which could be improved later by numerical simulations, also able to tackle more elaborate situations.

We may imagine 1D-nanodoped material wave guided to a target.

Eq.(1.1) is a particular case of the relation[9]

$$\delta[\phi(\tau)] = \Sigma_n\, \delta(\tau-\tau_n)\, |\phi(\tau-\tau_n)|^{-1} \qquad (6.1)$$

in which the τ_n's the zeroes of $\phi(\tau)$.

When $\phi\nu(\tau) = J\nu(\tau)$ where $J\nu$ is the Bessel function of the first kind of order ν with the zeroes $j\nu,_n$ n = 1,2…. we get from (6.1)

$$\delta[J_0(\tau)] = \Sigma_n\, \delta(\tau-j_{0,n})\, |J_1(j_{0,n})|^{-1} \quad a)$$
$$\delta[J_\nu(\tau)] = \Sigma_n\, \delta(\tau-j_{\nu,n})\, |J_{\nu-1}(j_{\nu n})|^{-1}\ \nu \geq 1 \quad b) \qquad (6.2)$$

For instance, proceeding as in Sec.(5,2), we get with (6.2a)

$$E\theta(r, z, t) = H\theta(r, z, t) = 0$$
$$\{E_r(r, z,t), H_r(r, z, t)\} = \{\underline{E}, \underline{H}\}\exp(ikz)\, J_1(kr)\, \partial_t \delta[J_0(\tau)]$$
$$\{E_z(r, z,t), H_z(r, z, t)\} = -i\,\{\underline{E}, \underline{H}\}\exp(ikz)\, J_0(kr)\, \partial_t \delta[J_0(\tau)] \qquad (6.3)$$

These solutions represent plane waves flashing at times $t = j_{0,n}t_{0,}$ n = 1,2…

The relation (1.1) supplies 1D-nanodoping made of delta Dirac pulses along a direction, Its generalization

$$\pi^2\delta[\sin(\pi x/x_0)]\, \delta[\sin(\pi y/y_0)] = x_0 y_0\, \Sigma_{m,n}\, \delta(x-mx_0)\, \delta(y-ny_0) \qquad (6.4)$$

would give a 2D-nanodoping with delta Dirac pulses at the vertices of a rectangular grid.

Using the relation[9] $\delta(x)\delta y = \delta(r)/\pi r$, $r = (x^2+y^2)^{1/2}$, the right hand side of (6.4) becomes $x_0 y_0\, \Sigma_{m,n}\, \delta r_{m,n} / \pi r_{m,n}$ with $r_{m,n} = [(x-mx_0)^2 + (y-ny_0)^2]^{1/2}$.

This result could be applied to flashes in the (x, ct) plane and generalized to 3D-nanodoping.

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