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Isothermal spheres and charged dust

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Abstract

The density profiles of dark matter halos are often modeled by an approximate solution to the isothermal Lane-Emden equation with suitable boundary conditions at the origin. It is shown here that such a model corresponds to an exact solution of the Einstein-Maxwell equations for charged dust.

Key Words

Lane-Emden equation; Isothermal sphere; Charged dust.

INTRODUCTION

The standard ΛCDM has as its principal matter component collisionless cold dark matter of an unknown nature. The rotation curves of spiral galaxies as well as the inferred mass of galaxy clusters are best explained by the existence of dark matter that dominates their mass content. Relatively recent work on colliding galaxy clusters appear to confirm this supposition[1,2]. Other possibilities, such as modifying Newton's equations or postulating a change in the gravitational interaction between dark and normal matter have been proposed, but have not gained favor.

In the case of the rotation curves of galaxies, the density distribution of dark matter is generally assumed to be spherical and to have an isothermal equation of state; i.e., a polytropic equation of state ($P = K\rho^{\gamma}$) where $\gamma = 1$. The hydrostatic balance equation may then be integrated to yield

$$\rho = \rho_0 \exp\left(-\frac{\Phi}{K}\right) \tag{1}$$

where Φ is the gravitational potential. Φ/K must then be a solution of the isothermal Lane-Emden equation. Non-singular solutions can be obtained by imposing appropriate boundary conditions, such as requiring that the solution and its first derivative vanish at the origin. The result is an exponential solution for the density of the form

$$\rho(r) = \frac{\rho_0}{\exp\left(\frac{\Phi}{K}\right)} \tag{2}$$

The isothermal Lane-Emden equation cannot be solved analytically and consequently Φ/K is expanded in a power series. The requirement that the first derivative vanish at the origin limits the expansion to even powers starting with $(\Phi/K)^2$. Expanding the exponential in the denominator of Eq. (2), keeping only the first two terms, and using the coefficient given by Chandrasekhar[3] for the leading $(\Phi/K)^2$ term results in the often used expression for the dark matter density,

$$\rho(r) \simeq \rho_0 \frac{r_0^2}{r_0^2 + r^2} \tag{3}$$

where $r_0 = \sqrt{\frac{6K}{4\pi G\rho_0}}$. It will be shown that the right hand side of this approximate expression corresponds to an *exact* solution of the coupled Einstein-Maxwell equations for exotic charged dust. Note that if r_0 is to be identified with the King radius, the numerical factor of 6 should be replaced by 9.

CHARGED DUST

The form of the metric for charged dust was introduced by Majumdar[4] and Papapetrou[5]. It is spherically symmetric and static, and can be motivated by considering the Reissner-Nordström metric

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2$$
$$- dr^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{4}$$

Assume the extreme form of this metric where $|Q| = m$, and introduce the isotropic coordinates $\bar{r} = r - m$. Doing so results in the metric

$$ds^2 = f^2\, dt^2 - f^{-2}\left[d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{5}$$

where $f = \left(1 + \frac{m}{\bar{r}}\right)^{-1}$. Henceforth the bar above the r will be dropped with the understanding that isotropic coordinates are used in what follows.

Using Newtonian mechanics and classical electrostatics, it is straightforward to show that a system of charged particles of mass m_i and charge q_i, where all of the particles have the same sign charge, will be in static equilibrium if $|q_i| = G^{1/2}m_i$. For a continuous distribution of mass ρ and charge σ, there will be equilibrium everywhere if $|\sigma| = G^{1/2}\rho$. This is what is known as charged dust. It has a general relativistic analog that was discovered by Papapetrou and Majumdar. Although spatial symmetry is not required, spherical symmetry will be assumed here.

Note, however, that the extremal condition $q = G^{1/2} m$ means that if q is chosen to be the minimal charge of one electron or 10^{-19} coulomb, then there is a minimal mass of $\sim 3.6 \times 10^{-9}$ kilogram giving a charge to mass ratio of 2.7×10^{-11}. This minimal mass is unusual in that it is very close in value to the reduced Plank mass of $\sqrt{\frac{\hbar c}{8\pi G}} = 4.3 \times 10^{-9}$ kilogram (much greater than the supersymmetric extension of the standard model predicting WIMPs having a mass of ~ 100 Gev/c^2).

The equilibrium of charged dust in general relativity has been treated extensively by W.B. Bonnor and others since the early 1960s. It is his paper on the equilibrium of a charged sphere[6] that forms the embarkation point for the work here[7]. The Einstein and Maxwell field equations applied to the metric of Eq. (5) show that the Newtonian condition for equilibrium given above must also hold in general relativity. In what follows, the charge will be chosen to be positive.

Bonnor obtained the equation that relates the general form of f to the density,

$$ff'' - 2f'^2 + \frac{2}{r} ff' - 4\pi\rho = 0 \tag{6}$$

Unfortunately, this equation is completely intractable unless $\rho = 0$, and as put by Lemos and Zanchin, "It is not a method for solving the differential equation of the Majundar-Papapetrou problem, it is an art of correct guessing[8]." In other words, one is reduced to guessing a form for the function f and hoping that the equation yields a physically meaningful density distribution.

The problem addressed by Bonnor was to find the density distribution of charged dust within a finite sphere of radius a that would match to the vacuum Reissner-Nordström solution at the boundary. This was successfully achieved using the following expression for f

$$f(r) = \left(a^3 + mr^2\right)^{\frac{1}{2}}(a + m)^{-\frac{3}{2}} \tag{7}$$

In Eq. (7), m is the mass of the charged dust contained within $r = a$. The density was found to be

$$\rho = \frac{3m}{4\pi a^3}\left(1 + \frac{m}{a}\right)^{-3}\left(1 + \frac{mr^2}{a^3}\right)^{-1} \tag{8}$$

The question addressed here is whether it is possible to find a function $f(r)$ that would result in a radially unlimited density distribution matching that given in Eq. (3) for dark matter. Indeed one can. Substitution of

$$f(r) = \sqrt{\frac{4}{3}\pi\rho_0}\left(a^2 + r^2\right)^{\frac{1}{2}} \tag{9}$$

into Eq (6) yields

$$\rho(r) = \rho_0\, \frac{a^2}{a^2 + r^2} \tag{10}$$

where a is now a free constant. This has the same form as Eq. (3) except that now the equality is exact and $\rho(r)$ is derived from a solution of the Einstein-Maxwell field equations. This is somewhat surprising given that the origins to Eq. (3) and Eqs. (9) and (10) are so different.

Both the isothermal sphere and the corresponding solution given here to the Einstein-Maxwell field equations are unrealistic in the context of dark matter since the total mass, proportional to r^2 at large radii, is infinite. More realistic models can be obtained by modifying these solutions, but this is outside the scope of this note.

There are also other solutions to Eq. (6) for the density distribution that exhibit unusual profiles that could prove useful. They result from the form of f given by

$$f(r) = \left(a + br^n\right)^{\frac{1}{2}}\left(1 - \frac{1}{1 + cr}\right), \qquad n = 1, 3/2 \tag{11}$$

for certain ranges of the constants a, b, and c. Figure 1 shows a density plot for two of these solutions.

Note that for $n = 3/2$ the density vanishes at $r = 0$, peaks, and then rises again (peaking to $\rho \sim 0.028$ at $r \sim 15$) before it decreases slowly to zero at infinity. There is observational support for such dark energy distributions—where, however, the density does not vanish at the origin—in galactic clusters[9]. Other solutions to Eq. (6) may exist that more closely match this observation. For $n = 1$, on the other hand, after the initial peak the density decreases monotonically.

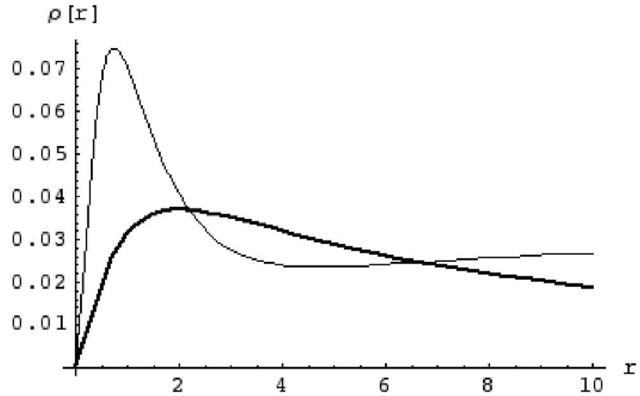

Figure 1 : A plot of the density distribution as a function of r given by Eq. (11) for $n = 1$ (dark) and $n = 3/2$ (light) for $a = b = c = 1$.

If one tries to generalize the solution of Eq. (9) to

$$f(r) = \sqrt{\frac{4}{3}\pi\rho_0}\left(a^\gamma + r^\gamma\right)^{\frac{1}{2}} \tag{12}$$

the solution for the density is found to be

$$\rho(r) = \rho_0 \frac{r^{\gamma-2}\,\gamma\left[2a^\gamma(1+\gamma) - r^\gamma(\gamma-2)\right]}{12\left(a^\gamma + r^\gamma\right)} \tag{13}$$

For $\gamma > 2$, the density has negative values for some range of r. Consequently, one must impose the condition that $\gamma \leq 2$. The plot of the density for $\gamma = 2$ and $\gamma < 2$ is shown in Figure 2. The cusp in the density for $\gamma < 2$ is a result of the density being singular at $r = 0$. Singular density functions are often used to model the mass distributions in elliptical galaxies. This is possible because the total mass as a function of r is finite[10]. This also the case for the density function given by Eq. (13).

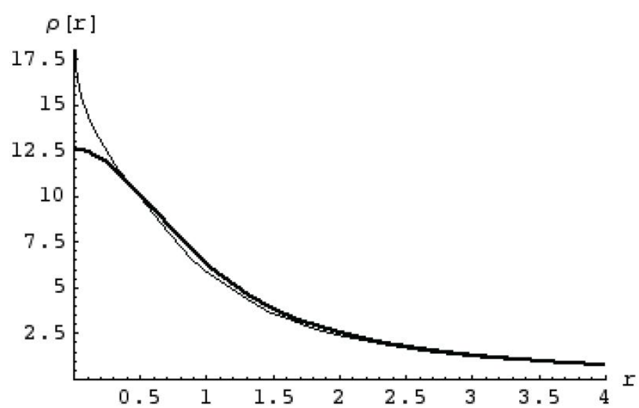

Figure 2 : A plot of the density distribution as a function of r given by Eq. (13) for $\gamma = 2$ (dark) and $\gamma = 1.9$ (light) for $\rho_0 = a = 1$. The density for $\gamma = 1.9$ is singular at $r = 0$.

Note that in computing the total mass the gravitational potential energy must also be included. This implies that one does not use the proper volume element in carrying out the integration of the density[11]; i.e.,

$$M(r) = 4\pi\int_0^r \rho_0 \frac{r^{\gamma-2}\,\gamma\left[2a^\gamma(1+\gamma) - r^\gamma(\gamma-2)\right]}{12\left(a^\gamma + r^\gamma\right)} r^2 \mathrm{d}r = \frac{1}{3}\pi\gamma r$$
$$\times\left(3a^\gamma\gamma - \frac{r^\gamma(\gamma-2)}{\gamma+1} - 3\gamma a^\gamma\, \mathit{Hypergeometric2F1}\left[\frac{1}{\gamma},1,1+\frac{1}{\gamma},-a^{-\gamma}r^\gamma\right]\right) \tag{14}$$

Given the result above for the density distribution, it is not surprising that for $\gamma > 2$, the total mass inside a radius r becomes negative as r increases, so that γ must again be limited to values less than or equal to 2.

Static charged dust solutions to the Einstein-Maxwell equations are known to be neutrally stable, meaning that they neither collapse or expand without external force being applied[12,13]. And while they are outside the scope of this paper, there are also stationary, axially symmetric, rigidly rotating solutions to the Einstein-Maxwell equations for charged dust[14]. Some of these are force-free[15] in the sense that the current generated by the velocity of the charged dust is parallel to the magnetic field generated by the motion.

SUMMARY

It has been shown that the approximate solution to the isothermal Lane-Emden equation, often used—with suitable boundary conditions at the origin—to model dark matter halos, corresponds to an exact solution of the Einstein-Maxwell equations for charged dust.

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