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Homomorphism and isomorphism of hilbert algebras in BCK-algebra

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ABSTRACT

The notion of BCK-algebras was formulated first in 1966 by K. Iséki Japanese Mathematician. In this paper we will discuss Homomorphism and Isomorphism Hilbert Algebras in BCK-algebras and its proposition.

KEYWORDS

BCK-algebra;
Hilbert algebras;
Homomorphism;
Isomorphism.

INTRODUCTION

BCK-algebra is originated from two different ways. One of the motivation is based on set theory, another motivation is from classical and non-classical propositional calculi. Here we will discuss Homomorphism and Isomorphism Hilbert Algebras in BCK-algebras and its proposition.

DEFINITION OF HOMOMORPHISM AND ISOMORPHISM

Definition

Suppose $(H;\to,1)$ and $(H';\to',1')$ are two Hilbert Algebras in BCK-algebras. A mapping $f:H\to H'$ is called a homomorphism from H into H', if for any $x,y\in H$,

$\mathbf{f(y\to x)=f(y)\to f(x)}$.

Definition

Suppose $(H;\to,1)$ and $(H';\to',1')$ are two Hilbert Algebras in BCK-algebras. A mapping $f:H\to H'$ is called a homomorphism from H into H', if for any $x,y\in H$, $f(y\to x)=f(y)\to f(x)$, and $f(H)=H'$, $F(H)=\{f(x):x\in H\}$, then f is called an epimorphism. If f both epimorphism and one-to-one, then f is called isomorphism.

In case $H=H'$ a homomorphism is called an endomorphism and an isomorphism is referred as an automorphism.

The set of all homomorphism from H into H' is denoted by $Hom(H,H')$,usually $Hom(H,H')\neq\phi$, because it contains the one homomorphism: $1:H\to H'$.

For any $f\in Hom(H,H')$, and any empty subset $H_1\subseteq H$, the set

$\mathbf{f^{-1}(H_1)=\{x\in H:f(x)\in H_1\}}$

Called the inverse image of H_1 under f.

In particular, $f^{-1}(\{1'\})$ is called the kernel of f. Note $f^{-1}(\{1'\})=\{x\in H: f(x)=1'\}$.

Theorem

Suppose $f: H \to H'$ is a homomorphism, then

(1) $f(1)=1'$,

(2) f is isotone.

Proof

Because $f(1)=f(1\to 1)=f(1)\to f(1)=1'\to 1'=1'$, (1) holds.

If $x, y \in H$, and $x \le y$, then $y \to x = 1$, by(1)

$\mathbf{f(y \to x) = f(y) \to f(x) = f(1) = 1'}$,

hence $f(x) \le f(y)$, proving(2).

Theorem

Suppose $(H; \to, 1)$ and $(H'; \to', 1')$ are two Hilbert Algebras in BCK-algebras. Let H_1' be an ideal of H', then for any $f \in Hom(H, H')$, $f^{-1}(H_1')$ is an ideal of H.

Proof

By theorem1.1(1), $1 \in f^{-1}(H_1')$. Assume that $y \to x \in f^{-1}(H_1')$, and $y \in f^{-1}(H_1')$, then

$\mathbf{f(y) \to f(x) = f(y \to x) \in f^{-1}(H_1'), f(y) \in f^{-1}(H_1')}$.

It follows that $f(x) \in H_1'$, so $x \in f^{-1}(H_1')$. This say that $f^{-1}(H_1')$ is an ideal of H.

Since $\{1'\}$ is an ideal of H', we have

Theorem

$Ker(f)$ is an ideal of H.

Definition

Suppose H is a Hilbert Algebras in BCK-algebras, a proper ideal H_1 of H is called obstinate, if for any $x, y \in H$, $x, y \notin H_1$, implies $y \to x \in H_1$, $x \to y \in H_1$.

Theorem

Suppose H is a Hilbert Algebras in BCK-algebras, H_1 is an ideal of H, the following are equivalent:

(1) H_1 is obstinate,

(2) H_1 is positive implicative and maximal,

(3) H_1 is implicative and maximal.

Theorem

Suppose H is a Hilbert Algebras in BCK-algebras, H_1 is an ideal of H, the following are equivalent:

(1) H_1 is obstinate,

(2) H_1 is maximal,

(3) H_1 is Prime

(4) H_1 is irreducible.

Theorem

Suppose H and H_1 are two Hilbert Algebras in BCK-algebras, H_1 is a proper ideal of H, then for any Hilbert Algebras in BCK-algebras H' there exists $f \in Hom(H, H')$ such that $Ker(f) = H_1$ if and only if H_1 is obstinate.

Proof

Suppose H_1 is obstinate, we define

$$\mathbf{f(x) = \begin{cases} 1' & x \in H_1 \\ a & x \in H - H_1 \end{cases}}$$

where a is any fixed element of H_1, and $a \ne 1'$, In order to $f \in Hom(H, H')$.

If $x, y \in H_1$, then $y \to x \in H_1$ as $y \to x \le x$, hence $f(y \to x) = 1'$. On the other hand

$\mathbf{f(y) \to' f(x) = 1' \to' 1' = 1'}$,

Therefore $f(y \to x) = f(y) \to' f(x)$.

If $x, y \notin H_1$, then $y \to x \in H_1$, because H_1 is obstinate, and so $f(y \to x) = 1'$. On the other hand

$\mathbf{f(y) \to' f(x) = a \to' a = 1'}$,

It follows that $f(y \to x) = f(y) \to' f(x)$.

If $x \notin H_1$, $y \in H_1$, then $y \to x \notin H_1$, and so

$\mathbf{f(y \to x) = a = 1' \to' a = f(y) \to' f(x)}$.

If $x \in H_1$, $y \notin H_1$, then $y \to x \in H_1$ as $y \to x \le x$, hence

$\mathbf{f(y \to x) = 1' = a \to' 1' = f(y) \to' f(x)}$.

Summarizing all the above we know $f \in Hom(H, H')$, and $Ker(f) = f^{-1}(1') = H_1$.

Conversely, suppose that for any Hilbert Algebras in BCK-algebras $(H'; \to', 1')$, there exists $f \in Hom(H, H')$ such that $Ker(f) = H_1$.

Assume $H' = \{1', a\}$, in which $\to$ is given by

$1' \to a = a$, $a \to a = a \to 1' = 1' \to 1' = 1'$.

then $(H'; \to, 1')$ is a Hilbert Algebras in implicative BCK-algebras.

By the hypothesis there exists $f \in Hom(H, H')$ such that $Ker(f) = H_1$, then $f^{-1}(a) = H - H'$.

For any $x, y \in H - H'$, we have $f(x) = f(y) = a$.

so

$f(y \to x) = f(y) \to' f(x) = a \to' a = 1'$,

$f(x \to y) = f(x) \to' f(y) = a \to' a = 1'$

This shows that $y \to x \in H_1$, $x \to y \in H_1$, hence H_1 is obstinate.

Theorem

Suppose X, Y, Z are three Hilbert Algebras in BCK-algebras, let $h : X \to Y$ be an epimorphism and $g \in Hom(X, Z)$. If $Ker(h) \subseteq Ker(g)$, then there exists a unique homomorphism $f : Y \to Z$ such that $f \bullet h = g$.

Proof

For any $y \in Y$, there is $x \in X$, such that $y = h(x)$.

For x, put $z = g(x)$, $y = h(x_1) = h(x_2), x_1, x_2 \in X$, then

$h(x_2 \to x_1) = h(x_2) \to h(x_1) = 1$,

so $x_2 \to x_1 \in Ker(h)$.

Since $Ker(h) \subseteq Ker(g)$, then $1 = g(x_2 \to x_1) = g(x_2) \to g(x_1)$.

Similarly, we obtain $g(x_1) \to g(x_2) = 1$, therefore $g(x_1) = g(x_2)$, this show that f is well-defined, and $y = h(x), z = g(x)$ and $f : y \mapsto z$, imply $g(x) = f(h(x))$.

Let $y_1, y_2 \in Y$, for any $x_1, x_2 \in X$, such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We have

$f(y_2 \to y_1) = f(h(x_2) \to h(x_1))$

$= f(h(x_2 \to x_1))$

$= g(x_2 \to x_1)$

$= g(x_2) \to g(x_1)$

$= f(h(x_2)) \to f(h(x_1))$

$= f(y_2) \to f(y_1)$

Hence $f \in Hom(Y, Z)$.

HOMOMORPHISM THEOREM

Definition

Suppose H and H_1 are two Hilbert Algebras in BCK-algebras, then there exists an epimorphism $f : H \to H'$, then we call H to be homomorphic to H_1, written $H \sim H'$; if there exists an isomorphism $f : H \to H'$, then we call H to be isomorphic to H_1, written $H \cong H'$.

Propositions

(1) $H \cong H$,

(2) If $H \cong H'$, then $H' \cong H$,

(3) If $H_1 \cong H_2$ and $H_2 \cong H_3$, then $H_1 \cong H_3$.

Theorem

Suppose H is a Hilbert Algebras in BCK-algebras, if H_1 is an ideal of H, then the quotient algebra H/H' is a homomorphic image of.

Proof

Let $f : H \to H/H'$, because $H/H' = f(H)$. then $H \sim H/H'$.

Theorem

(Homomorphism Theorem)Suppose H and H_1 are two Hilbert Algebras in BCK-algebras, if $f : H \to H'$ is an epimorphism then $H/Ker(f)\cong H_1$.

Proof

Because $Ker(f)$ is an ideal of H, then $H/Ker(f)$ is a Hilbert Algebras in BCK-algebras, and $C_1 = Ker(f)$.

Assume $\mu: H/Ker(f)\to H$ and $\mu(C_x) = f(x)$. In the following we proof that μ is an isomorphism.

If $C_x = C_y$, then $y\to x, x\to y\in Ker(f)$, so

$\mathbf{f(y\to x)=f(x\to y)=1}$,

$\mathbf{f(y)\to f(x)=f(x)\to f(y)=1}$.

By BCI-4we have $f(x)= f(y)$ and $\mu(C_x)=\mu(C_y)$. This shows that μ is a mapping from $H/Ker(f)$ to H'.

For any $y\in H'$, there is $x\in H$, such that $y = f(x)$, so

$\mathbf{\mu(C_x)=f(x)=y}$,

hence $\mu: H/Ker(f)\to H$.

If $C_x \neq C_y$, then x, y do not belong to the same equivalent class. Thus $x\to y\notin Ker(f)$ or $y\to x\notin Ker(f)$.

Suppose $y\to x\notin Ker(f)$, then $f(y)\to f(x)= f(y\to x)\neq 1$.

So $f(x)\neq f(y)$. This says that μ is one-to-one.Since

$$\mathbf{\mu(C_y\to C_x)=\mu(C_{y\to x})=f(y\to x)}$$
$$\mathbf{= f(y)\to f(x)=\mu(C_y)\to\mu(C_x)},$$

so μ is a homomorphism, putting the above facts together we know that μ is an isomorphism from $H/Ker(f)$ to H'.

Theorem

If $f: H\to H'$ is an epimorphism, then the following are equivalent:

(1) $Ker(f)$ is a commutative(positive implicative, implicative) ideal,

(2) $\{1\}$ is a commutative (positive implicative, implicative) ideal of H',

(3) H' is a Hilbert Algebras in commutative(positive implicative, implicative) BCK-algebras.

Proof

(2) $\Leftrightarrow$ (3) Because $Ker(f)$ is a commutative ideal of H is equivalent to $H/Ker(f)$ being a Hilbert Algebras in commutative BCK-algebras. Suing Homomorphism Theorem we obtain $H/Ker(f)\cong H'$, and so $(1)\cong(3)$.

Theorem

If $f: H\to H'$ is an epimorphism, if H is bounded (commutative, positive implicative, implicative), then so is H'.

Proof

If H is bounded (commutative, positive implicative, implicative), then so is $H/Ker(f)$. Since $H/Ker(f)\cong H'$, by Homomorphism Theorem, H' is bounded (commutative, positive implicative, implicative).

Theorem

If $f: H\to H'$ is an epimorphism, and H_2 is an ideal of H', then $H/H_1\cong H'/H_2$ and $H_1 = f^{-1}(H_2)$.

Proof

The natural homomorphism from H' to H'/H_2 is denoted by v, then $\mu = v\bullet f$ is an

Epimorphism from H to H'/H_2, we now prove $Ker(\mu)= f^{-1}(H_2)$.

For any $x\in H$, then $\mu(x)=(v\bullet f)(x)=v(f(x))=C_{f(x)}$.

where $C_{f(x)}$ is the equivalent class containing $f(x)$ in H'/H_2. Suppose $y\in f^{-1}(H_2)$,

then $f(y)\in H_2$, so $C_{f(y)} = H_2$. This says $\mu(y)=H_2$, hence $y\in Ker(\mu)$, thus we obtain

$f^{-1}(H_2)\subseteq Ker(\mu)$.

Let $x \in Ker(\mu)$, then $\mu(x) = H_2$. Combining $\mu(x) = C_{f(x)}$, we have $C_{f(x)} = H_2$.It follows that $f(x) \in H_2$, and so $x \in f^{-1}(H_2)$. This means that

$f^{-1}(H_2) \supseteq Ker(\mu)$.

Therefore $f^{-1}(H_2) = Ker(\mu)$.

by Homomorphism Theorem $H / Ker(\mu) \cong H' / H_2$

Hence $H/H_1 \cong H'/H_2$.

Theorem

If $f : H \to H'$ is an epimorphism and $R \in \Re(H)$. If $Ker(f) \subseteq R$, then $f^{-1}(f(R)) = R$.

Proof

Obviously $R \subseteq f^{-1}(f(R))$.Assume $x \subseteq f^{-1}(f(R))$, then $f(x) \in f(R)$.Thus there is $y \in R$, such that $f(x) = f(y)$, so

$\mathbf{f(y \to x) = f(y) \to f(x) = 1}$

Hence $\mathbf{y \to x \in Ker(f) \subseteq R}$.

Noticing that $R \in \Re(H)$, then $x \in R$. Therefore $f^{-1}(f(R)) \subseteq R$, hence $f^{-1}(f(R)) = R$.

Theorem

Suppose H is a Hilbert Algebras in BCK-algebras, H_1, H_2 are two ideal of H and $H_2 \subseteq H_1$, let $v : H \to H / H_2$ and $\mu : H / H_2 \to (H / H_2)/(H_1 / H_2)$ be natural homomorphism, then

$\mathbf{H / H_1 \cong (H / H_2)/(H_1 / H_2)}$.

Proof

Let $f = \mu \bullet f$, then f is an epimorphism from H to $(H / H_2)/(H_1 / H_2)$. Hence

$\mathbf{H / Ker(f) \cong (H / H_2)/(H_1 / H_2)}$.

Since

$\mathbf{Ker(f) = \{x \in H : f(x) = H_1 / H_2\}}$,

so $Ker(f) = f^{-1}(f(H_1)) = H_1$.

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