Int. J. Chem. Sci.: 13(4), 2015, 1655-1665
ISSN 0972-768X
www.sadgurupublications.com

GROUP THEORETIC ORIGINS OF CERTAIN GENERATING FUNCTIONS OF LEGENDRE POLYNOMIALS

P. L. RAMA KAMESWARI and V. S. BHAGAVAN[*]

Department of Mathematics, K. L. University, VADDESWARAM – 522502, Dist.: Guntur (A.P.) INDIA
Department of Mathematics, Swarnandhra College of Engg. & Tech., Seetharampuram, NARSAPURAM – 534280, Dist.: West Godawari (A.P.) INDIA

ABSTRACT

In this paper, Weisner's group theoretic method is utilized to obtain the generating functions for the Legendre polynomials P_n (x). To derive the elements of Lie algebra, a suitable interpretation to the index n is given. Further, a linear independent differential operators was derived, which generates a Lie algebra. The principle interest in our results lies in the fact that, how the Weisner's group-theoretic method can be applied suitably to the Legendre polynomials in order to derive six generating functions. Many results obtained are well known but some of them are believed to be new in the theory of special functions. Mathematics Subject Classification (2010): 33C10, 33C45, 33C50, 33C80.

Key words: Special functions, Legendre polynomials, Group theoretic method, Generating functions.

INTRODUCTION

Group theoretic method was proposed by Louis Weisner in the year 1955 and he employed this method to find generating relations for a large class of special functions. Weisner discussed the group theoretic significance of generating functions for hypergeometric functions namely Hermite, Bessel functions etc. This technique was used by Khanna and Bhagavan[1], Khan and Pathan[2] for obtaining generating functions for various special functions. The importance of group theoretic method is to create a connection between special functions and matrix groups and plays a very important role in constructing the first order linear differential operators, which generates a Lie algebra that is isomorphic to some matrix Lie algebra. Miller[3], Mc Bride[4], and Srivastava and Manocha[5] reported group theoretic method for obtaining generating relations in their books. Hypergeometric polynomials/hypergeometric series in one and more variables arise naturally and rather frequently in a wide variety of problems in applied mathematics, theoretical physics,

[*]Author for correspondence; E-mail: ramaravikumar.i@gmail.com; drvsb002@kluniversity.in

engineering sciences, statistics and operation research etc., In fact, a considerable field of physical and quantum mechanical situations (such as Schrodinger's wave mechanics) and various types of distributions in probability theory lead naturally to such classical orthogonal polynomials as the Legendre and Laguerre polynomials.

The principle objective of this paper is to derive some more interesting bilateral (or bilinear) generating relations for P_n (x) using Weisner[1] group-theoretic method[4] by giving an interpretation to the index n. The usefulness of this method is that it yields a set of six generating relations.

Definition

The Legendre polynomials P_n (x) satisfy the following descending and ascending recurrence relations, respectively:

$$\left(1-x^2\right)\frac{d}{dx}P_n(x)=n\left[P_{n-1}(x)-x\,P_n(x)\right] \qquad \ldots(1)$$

and

$$\left(1-x^2\right)\frac{d}{dx}P_n(x)=(n+1)\left[xP_n(x)-xP_{n+1}(x)\right] \qquad \ldots(2)$$

These two independent differential recurrence relations determine the linear ordinary differential equation

$$\left(1-x^2\right)D^2P_n(x)-2x\;DP_n(x)+n\;(n+1)P_n(x)\;=\;0 \qquad \ldots(3)$$

where $D=\frac{d}{dx}$. The proofs of these results are obvious.

Group – Theoretic Discussion

Let us write the differential equation (3) in operator functional notation as

$$L\left(x,\frac{d}{dx},n\right)=\left(1-x^2\right)D^2-2\,x\;D+n\;(n+1)=0 \qquad \ldots(4)$$

In order to use Weisner's method, we now construct from (3) the following partial differential equation by replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\,\frac{\partial}{\partial x}$ and $P_n(x)$ by $u(x,y)$:

$$\left[\left(1-x^2\right)\frac{\partial^2}{\partial x^2}+y^2\frac{\partial^2}{\partial y^2}-2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}\right]u(x,y)=0 \qquad \ldots(5)$$

where $u(x,y)=y^n P_n(x)$ is a solution of (5).

Let L represents the partial differential operator of (5), given by

$$L\left(x,\frac{d}{dx},n\right)=\left[\left(1-x^2\right)\frac{\partial^2}{\partial x^2}+y^2\frac{\partial^2}{\partial y^2}-2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}\right] \qquad \ldots(6)$$

We now seek linearly independent lowering and raising differential operators B and C each of the form

$$A_1(x,y)\frac{\partial}{\partial x}+A_2(x,y)\frac{\partial}{\partial y}+A_3(x,y)$$

such that

$$B\left[P_n(x)\,y^n\right]=b_n\,P_{n-1}(x)\,y^{n-1}$$

$$C\left[P_n(x)\,y^n\right]=c_n\,P_{n+1}(x)\,y^{n+1} \qquad \ldots(7)$$

where b_n and c_n are expression in n which are independent of x and y.

Each A_i (x, y), i = 1, 2, 3, on the other hand, is an expression in x and y, which is independent of n.

This necessitates the bringing into use of the recurrence relations (1) and (2). With the help of (1) and (2), it follows from (7) that

$$C=\left(1-x^2\right)y\frac{\partial}{\partial x}-xy^2\frac{\partial}{\partial y}-xy$$

$$B=\left(1-x^2\right)y^{-1}\frac{\partial}{\partial x}+x\frac{\partial}{\partial y} \qquad \ldots(8)$$

To find the group of operators, let us write $A\equiv y\frac{\partial}{\partial y}$.

Then we have the operators A, B and C, which satisfy the following commutator relations:

$$[A, B] = -B \qquad [A, C] = C \qquad [B, C] = -2.4 - 1 \qquad \ldots(9)$$

Now, every linear differential operator of the first order generates a one parameter Lie group[4] and therefore, these commutator relations show that the set of operators {1, A, B, C} generate a three-parameter Lie group. Furthermore, we would like to prove that these operators commute with the partial differential operator L. We express L in terms of these operators.

We know that

$$Lu = \left(1 - x^2\right)\frac{\partial^2 u}{\partial x^2} + y^2\,\frac{\partial^2 u}{\partial y^2} - 2x\,\frac{\partial u}{\partial x} + 2y\,\frac{\partial u}{\partial y}.$$

and

$$CBu = \left(1 - x^2\right)\frac{\partial^2 u}{\partial x^2} - x^2 y^2\,\frac{\partial^2 u}{\partial y^2} - 2x\,\left(1 - x^2\right)\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} - 2x^2 y\,\frac{\partial u}{\partial y}$$

We get

$$\left[\left(1 - x^2\right)L - CB\right]u = A^2 u.$$

Therefore,

$$\left(1 - x^2\right)Lu = \left(CB + A^2\right)u \qquad \ldots(10)$$

By using the commutator relations, we prove that the operators A, B and C commute with $(1 - x^2)$ L and hence with $R = r_1 A + r_2 B + r_3 C + r_4$, where each r_i $(i = 1, 2, 3, 4)$ is an arbitrary constant, R is the set of differential operators.

This Lie algebra determines a root system and Weyl group. The extended form of the group generated by each of the operators A, B and C as follows:

$$e^{aA} f\,(x, y) = f\,\left(x, e^a y\right) \qquad \ldots(11)$$

$$e^{bB} f\,(x, y) = f\left(\frac{xy + b}{\sqrt{y^2 + 2\,bxy + b^2}}, \sqrt{y^2 + 2\,bxy + b^2}\right) \qquad \ldots(12)$$

$$e^{cC} f(x, y) = \frac{1}{\sqrt{c^2y^2 + 2\,cxy + 1}}\, f\left(\frac{x + cy}{\sqrt{c^2y^2 + 2\,cxy + 1}}, \frac{y}{\sqrt{c^2y^2 + 2\,cxy + 1}}\right) \quad \ldots(13)$$

where a, b and c are arbitrary constants and f (x, y) is an arbitrary function.

Then it is evident that

$$e^{cC} e^{bB} f(x, y) = \frac{1}{\sqrt{c^2y^2 + 2\,cxy + 1}}\, f(\xi, \eta) \quad \ldots(14)$$

Where

$$\xi = \frac{xy + cy^2 + b\left(c^2y^2 + 2\,cxy + 1\right)}{\sqrt{\left[y^2 + b^2\left(c^2y^2 + 2\,cxy + 1\right) + 2b\left(xy + cy^2\right)\right]\left(c^2y^2 + 2\,cxy + 1\right)}}$$

$$\eta = \sqrt{\frac{y^2 + b^2\left(c^2y^2 + 2\,cxy + 1\right) + 2b\left(xy + cy^2\right)}{c^2y^2 + 2\,cxy + 1}}$$

Generating Functions

Derivation from the Operator (A – υ)

We see that A generates a trivial group. Say $u(x, y) = y^{\upsilon} P_{\upsilon}(x)$ is a solution of the simultaneous equations Lu = 0 and (A – υ) = 0 for arbitrary υ. Therefore, we determine generating functions of $P_n(x)$ by finding $e^{bB + cC}\,[y^{\upsilon} P_{\upsilon}(x)]$.

We need to consider three cases.

Case-1: Suppose b = 1, c = 0. Since for an arbitrary function f (x, y)

$$e^{B} f(x, y) = f\left(\frac{xy + 1}{\sqrt{1 + 2\,xy + y^2}}, \sqrt{1 + 2\,xy + y^2}\right)$$

We find

$$e^{B} f(x, y) = \left(1 + 2\,xy + y^2\right)^{\frac{\upsilon}{2}} P_{\upsilon}\left(\frac{1 + xy}{\sqrt{1 + 2\,xy + y^2}}\right) \quad \ldots(15)$$

Since $B[y^{\upsilon}P_{\upsilon}(x)] = \upsilon y^{\upsilon-1} P_{\upsilon-1}(x)$, we have

$$e^{B}\left[y^{\upsilon}P_{\upsilon}(x)\right] = \sum_{n=0}^{\upsilon} \frac{(-1)^{n}(-\upsilon)_{n}}{n!} y^{\upsilon-n} P_{\upsilon-n}(x) \qquad \ldots(16)$$

Equating expressions (15) and (16) and replacing $-y^{-1}$ by t, we get –

$$\left(1-2xt+t^{2}\right)^{\frac{\upsilon}{2}} P_{\upsilon}\left(\frac{x-t}{\sqrt{1-2xt+t^{2}}}\right) = \sum_{n=0}^{\upsilon} \frac{(-\upsilon)_{n}}{n!} P_{\upsilon-n}(x)t^{n} \qquad \ldots(17)$$

Case-2: Suppose b = 0, c = 1. In this case, we have

$$e^{C}\left[y^{\upsilon}P_{\upsilon}(x)\right] = y^{\upsilon}\left(1+2xy+y^{2}\right)^{\frac{-\upsilon-1}{2}} P_{\upsilon}\left(\frac{x+y}{\sqrt{1+2xy+y^{2}}}\right) \qquad \ldots(18)$$

On the other hand,

$$e^{c}\left[y^{\upsilon}P_{\upsilon}(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^{n}(\upsilon+n)!}{\upsilon!n!} P_{\upsilon+n}(x)y^{\upsilon+n} \qquad \ldots(19)$$

Equating the expressions (18) and (19) and replacing –y by t′ we get –

$$\left(1-2xt+t^{2}\right)^{\frac{-\upsilon-1}{2}} P_{\upsilon}\left(\frac{x-t}{\sqrt{1-2xt+t^{2}}}\right) = \sum_{n=0}^{\infty} \frac{(\upsilon+n)!}{\upsilon!n!} P_{\upsilon+n}(x)t^{n} \qquad \ldots(20)$$

Case-3: Suppose bc ≠ 0, without any loss of generality we can choose b = − 1 and c = 1, so we have

$$\left(1+2xy+y^{2}\right)^{\frac{-\upsilon}{2}-\frac{1}{2}} P_{\upsilon}\left(\frac{-(xy+1)}{\sqrt{1+2xy+y^{2}}}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\upsilon} \frac{(-1)^{n}(-\upsilon)_{k}(\upsilon-k+1)_{n}}{k!n!} P_{\upsilon-k+n}(x)y^{\upsilon-k+n}$$

or

$$\left(1-2xt+t^{2}\right)^{\frac{-\upsilon}{2}-\frac{1}{2}} P_{\upsilon}\left(\frac{1-xt}{\sqrt{1-2xt+t^{2}}}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\upsilon} \frac{(-1)^{\upsilon-k}(-\upsilon)_{k}(\upsilon-k+1)_{n}}{k!n!} P_{\upsilon-k+n}(x)t^{\upsilon-k+n} \qquad \ldots(1)$$

Derivation from the Operator not Conjugate to (A-υ)

Let $S = e^{cC} e^{bB}$, where b and c are arbitrary constants

Now according to McBride we find that –

$$e^{bB}Ce^{-bB} = -2bA - b^2B + C - b$$

$$e^{bB}Ae^{-bB} = A + bB$$

$$e^{cC}Ae^{-cC} = A + cC$$

$$e^{cC}Be^{-cC} = 2cA + B - c^2C + c$$

Consider the set of linear differential operators

$\{R/R = r_1A + r_2B + r_3C + r_4$, for all combinations of zero and non-zero coefficients except for $r_1 = r_2 = r_3 = 0\}$.

We find that

$$SA\,S^{-1} = e^{cC}\,e^{bB}\,Ae^{-bB}e^{-cC}$$

$$= (1 + 2bc)\,A + bB - c(1 + bc)\,C + bc$$

Then

$$r_1 = 2bc \qquad r_2 = b \qquad r_3 = -c\,(1 + bc)$$

From two of these three equations we can find b and c in terms of r_1, r_2 and r_3. The third equation then imposes a restrictive relation on the r_i (i = 1, 2, 3), which $r_1^2 + 4\,r_2 r_3 = 1$.

Therefore, (A-υ) is not conjugate to operators for which $r_1^2 + 4\,r_2 r_3 = 0$.

We consider the following Cases :

Case-1: If $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, we seek a solution of the system Lu = 0 and $(B + \lambda)\,u = 0$.

where λ is a non-zero constant.

For convenience, we choose $\lambda = 1$ and write the equation as Lu = 0 and (B + 1)u = 0.

A solution of this system is –

$$u(x, y) = e^{xy}{}_0F_1\left(-\,;1\,;\frac{1}{4}y^2\left(x^2-1\right)\right)$$

If this function expanded in powers of y, we get

$$e^{xy}\left(-\,;1\,;\frac{1}{4}y^2\left(x^2-1\right)\right)=\sum_{n=0}^{\infty}\frac{P_n(x)y^n}{n!} \qquad \ldots(22)$$

Which can equally well be written in term of Bessel's function as

$$e^{xy}\ J_0\left(y\sqrt{1-x^2}\right)=\sum_{n=0}^{\infty}\frac{P_n(x)y^n}{n!}$$

Case-2: If $r_1 = 2$, $r_2 = 1$, $r_3 = -1$, we are led to this choice by considering e^{cC} $(B - \omega)$ e^{-cC}.

where is a non zero constant. We find that –

$$e^{cC}\,(B-\omega)\,e^{-cC} = 2\,cA + B + c^2C + (c-\omega)$$

If we let c = 1 then we determine the solution of the system

Lu = 0 and $(2A + B - C + 1 - \omega) = 0$. From the generating function of (22), by replacing y by –y we get –

$$u\,(x,-y) = e^{-x\omega\omega}{}_0F_1\left(-\,;1\,;\frac{1}{4}\,\omega^2y^2(x^2-1)\right)$$

We know that for an arbitrary function f (x, y).

$$e^{c}f\left(x, y\right)=\frac{1}{\sqrt{1+2xy+y^2}}\ f\left(\frac{x+y}{\sqrt{1+2\,xy+y^2}},\frac{y}{\sqrt{1+2\,xy+y^2}}\right)$$

Thus

$$e^{c}u\left(x,-\omega y\right)=\frac{1}{\sqrt{1+2\,xy+y^{2}}}\,\exp\left(\frac{-(x+y)\,\omega y}{\sqrt{1+2\,xy+y^{2}}}\right)\,{}_{0}F_{1}\left(-;1;\frac{1}{4}\frac{\omega^{2}y^{2}\,(x^{2}-1)}{\left(1+2\,xy+y^{2}\right)^{2}}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\left(1+2xy+y^{2}\right)^{-\frac{\upsilon}{2}-\frac{1}{2}}\left(-1\right)^{n}\omega^{n}y^{n}}{n\,!}\,P_{n}\left(\frac{x+y}{\sqrt{1+2xy+y^{2}}}\right)$$

With the help of (20), we get –

$$\frac{1}{\sqrt{1+2xy+y^{2}}}\exp\left(\frac{-\left(x+y\right)\omega y}{1+2xy+y^{2}}\right)\,{}_{0}F_{1}\left(-\,;1\,;\frac{1}{4}\frac{\omega^{2}y^{2}\left(x^{2}-1\right)}{\left(1+2xy+y^{2}\right)^{2}}\right)$$

$$=\sum_{n=0}^{\infty}L_{n}\left(\omega\right)P_{n}\left(x\right)y^{n} \qquad \ldots(23)$$

Which is a bilateral generating function for P_n (x).[6]

Case-3: Let $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. we seek a solution of the system

Lu = 0 and (c + Ø) u = 0, where Ø is a non-zero constant. We may avoid actually solving this system by noting that –

$$e^{bB}e^{cC}\,(B+1)\,e^{-cC}e^{-bB}=2c\left(1+bc\right)A+\left(1+bc\right)^{2}B-c^{2}C+c\left(1+bc\right)+1\,.$$

If we choose b = 1 and c = –1, we get

$$e^{B}e^{-C}\left(B+1\right)e^{C}e^{-B}=-C+1$$

Therefore we can obtain a solution of Lu = 0 and (C – 1) u = 0 by transforming the generating function (22) as –

$$e^{B}e^{-c}\left\{e^{xy}\,{}_{0}F_{1}\left(-;1;\frac{1}{4}y^{2}\left(x^{2}-1\right)\right)\right\}=\frac{1}{y}\exp\left(-1-\frac{x}{y}\right){}_{0}F_{1}\left[-;1;\frac{1}{4}\left(\frac{x^{2}-1}{y^{2}}\right)\right]$$

If we let -1/y = t and expand in powers of t we get –

$$(-t)\exp(-1+xt)\ {}_0F_1\left(-;1;\frac{1}{4}t^2\left(x^2-1\right)\right)=\frac{-t}{e}\sum_{n=0}^{\infty}\frac{P_n(x)t^n}{n!}$$

or

$$e^{xt}\ {}_0F_1\left(-;1;\frac{1}{4}t^2\left(x^2-1\right)\right)=\sum_{n=0}^{\infty}\frac{P_n(x)t^n}{n!}$$

ACKNOWLEDGMENT

The authors wish to express their sincere thanks to the referees for their valuable suggestions that resulted in the presentation of the paper in its current form.

REFERENCES

1. I. K. Khanna, V. Srinivasa Bhagavan and M. N. Singh, Generating Relations of the Hypergeometric Functions by the Lie Group-Theoretic Method, Math. Phys. Anal. Geometry, **3**, 287-303 (2000).
2. S. Khan, M. A. Pathan and G. Yasmin, Representation of a Lie Algebra G(0,1) and Three Variable Generalized Hermetic Polynomials, H_n (x, y, z), Integral Transforms Special Functions, **13**, 59-64 (2002).
3. W. Jr. Miller, Lie Theory and Special Functions: Academic Press, New York and London (1968).
4. E. Mc. Bride, Obtaining Generating Functions, Springer-Verlag, New York (1971).
5. H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted/ Wieley, New York (1984).
6. S. Khan and G. Yasmin, Lie-Theoretic Generating Relations of Two Variable Laguerre Polynomials, Rep. Math. Phys., **51**, 1-7 (2003).
7. L. Weisner, Group-Theoretic Origin of Certain Generating Functions, Pacific J. Math., **5**, 1033-1039 (1995).
8. L. Weisner, Generating Functions for Hermite Functions, Canad. J. Maths., **11**, 141-147 (1995).
9. L. Weisner, Generating Functions for Bessel functions, Canad. J. Maths, **11**, 148-155 (1995).

10. E. D. Rainville, Special functions, Academic press, New York (1968).

11. S. D. Bajpai and M. S. Arora, Ann. Math. Biaise. Pascal., **1**, 75-83 (1994).

Accepted : 25.09.2015