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GENERALIZED CANONICAL SINE TRANSFORM

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ABSTRACT

As generalization of the fractional Sine transform (FRST), the canonical sine transform (CST) has been used in several areas, including optical analysis and signal processing. Besides, the canonical sine transform is also useful for radar system analysis, filter design, phase retrieval pattern recognition, and many other verities of branches of mathematics and engineering. In this paper we have proved some important results about the analyticity theorem; we have also proved the Scaling property of canonical sine transform.

Keywords: Linear canonical transform, CCT, Fractional fourier transform.

INTRODUCTION

Integral transforms had provided a well establish and valuable method for solving problems in several areas of both Physics and Applied Mathematics. The roots of the method can be stressed back to the original work of Oliver Heaviside in 1890. This method proved to be of great importance, in the initial and final value problems for partial differential equations. Due to wide spread applicability of this method for partial differential equations involving distributional boundary conditions, many of the integral transforms are extended to generalized functions.

The idea of the fractional powers of Fourier operator appeared in mathematical literature as early in 1930. It has been rediscovered in quantum mechanics by Namias[9]. He had given a systematic method for the development of fractional integral transforms by means of eigenvalues. Later on numbers of integral transforms are extended in its fractional domain. For examples Almeida[2] had studied fractional Fourier transform, Akay[1] developed fractional Mellin transform, Pei, Ding[12] studied fractional cosine and sine transforms, etc. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

Recently further generalization of fractional Fourier transform known as linear canonical transform was introduced by Moshinsky[8] in 1971. Pei, Ding[6] had studied its eigen value aspect.

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Linear canonical transform is a three parameter linear integral transform which has several special cases as fractional Fourier transform, Fresnel transform, Chirp transform etc. Linear canonical transform is defined as,

$$[LCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot e^{-i(s/b)t} f(t)\, dt, \text{ for } b \neq 0$$

$$= \sqrt{d}\; e^{\frac{i}{2}(cd)s^2} \cdot f(d\cdot s), \text{ for } b = 0, \text{ with } ad - bc = 1,$$

where a, b, c, and d are real parameters independent on s and t.

Testing Function Space (I)

An infinitely differentiable complex valued function ϕ on R^n belongs to I(R^n), if for each compact set $I \subset s_\alpha$

where $s_\alpha = \{t :\in R^n, |t| \leq \alpha, \alpha > 0\}$ and for $I \in R^n$,

$$\gamma_{E,I.(\phi)} = \sup_{t \in I} \left|D^k\, \phi(t)\right| < \infty$$

Proposition

$K_s(t,s) \in I$, Where $K_s(t,s) = (-i)\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin(s/b)\, t$

Proof: We have to prove $\gamma_{I,K}\, K_s(t,s) < \infty$, is to show that

$$\sup\left|D_t^n\, K_s(t,s)\right| < \infty$$

We know that

$$D_t^n\, K_s(t,s) = D_t^n\left[C_1\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin(s/b)t\right]$$

Let $f = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin(s/b)t$

$$f' = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \cos(s/b)t\left(\frac{s}{b}\right). + \sin(s/b)t \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot i\left(\frac{a}{b}\right).t$$

$$f' = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left[\cos(s/b)t\left(\frac{s}{b}\right). + \sin(s/b)t \cdot i\left(\frac{a}{b}\right).t\right]$$

$$f'' = e^{\frac{i}{2}(a/b)t^2}\left[-\sin(s/b)t\,(s^2/b^2) + \cos(s/b)t\;\; i(as/b^2)t + i(a/b)\cdot\sin(s/b)t\right]$$
$$+\left(\cos(s/b)t\,(s/b) + \sin(s/b)t \quad i(a/b)t\right)e^{\frac{i}{2}(a/b)t^2}\cdot i(a/b)t$$

$$= e^{\frac{i}{2}(a/b)t^2}\left[\cos(s/b)t\;\; i(as/b^2)t - \sin(s/b)t\,(s^2/b^2) + i(a/b)\cdot\sin(s/b)t\right.$$
$$\left.+\cos(s/b)t\,(s/b)i(a/b)t - \sin(s/b)t\,(a^2/b^2)t^2\right].$$

and so on

$$F^n = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left[\cos(s/b)t\;C_1(s) + \sin(s/b)t\;\;C_2(s)\right],$$

Where $C_1(s)$ and $C_2(s)$ are functions of $'s$,

$$\therefore \left|f^n(t)\right| \le \left|e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\right|\left[\left|\cos(s/b)t\;C_1(s)\right| + \left|\sin(s/b)t\;C_2(s)\right|\right]$$

$$\therefore \left|f^n(t)\right| < \infty$$

$$\therefore \sup\left|D_t^n \quad K_s(t,s)\right| < \infty,$$

Hence $K_s(t,s) \in I(R^n)$

Generalized canonical sine transform (CST)

The canonical sine transform of $f \in I^1(R^n)$ can be defined by,

$$\{CST\;\; f(t)\}(s) = \langle f(t),\, K_s(t,s)\rangle.$$

Where $K_s(t,s) = \quad (-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot\sin(s/b)\;t$ …(1.1)

Clearly Kernel $K_s(t,s) \in$ I and $K_s(t,s) \in I^1(R^n)$ the kernel (1) Satisfies the following properties.

Properties of Kernel

(i) $\mathrm{K}_{(a,b,c,d)}\;(t,s) \ne \mathrm{K}_{(a,b,c,d)}\;(t,s)$, if $a \ne d$

(ii) $\mathrm{K}_{(a,b,c,d)}\;(t,s) = \mathrm{K}_{(a,b,c,d)}\;(t,s)$, if $a = d$

(iii) $\mathrm{K}_{(a,b,c,d)}\;(t,s) = \mathrm{K}_{(a,-b,-c,d)}\;(t,s)$

(iv) $\mathrm{K}_{(a,b,c,d)}\;(-t,s) = \mathrm{K}_{(a,b,c,d)}\;(t,-s)$

Analyticity theorem for canonical sine transform

Let $f \in \quad I^1(R^n)$ and let its canonical sine transform be defined by,

$$\{CST\ f(t)\}(s) = -\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot i\ \sin(s/b)t\ f(t)\,dt$$

then *{*CS*T f(t)} (s)* is analytic on C^n,

Proof: Let s*: {s_1, s_2,, ………s_j……s_n}* $\in s^n$

We first prove that,

$$\frac{\partial^k}{\partial s_j^k}\{CST\ f(t)\}(s) =< f(t); \frac{\partial^k}{\partial s_j^k} K_s(t,s) > . \qquad \ldots(1.2)$$

$$K_s(t,s) = -\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot i\ \sin(s/b)t,$$

We prove the result *k = 1*, the general result following by induction.

For fixed $s_j \neq 0$ choose two concentric circles P and P¹ with centre s_j and radii r and r_1 respectively such that $0<r<r_1<|s_j|$

Let Δs_j be a complex increment satisfying $0<\left|\Delta s_j\right|<r$

$$\frac{(CST)\ (s+\Delta s_j)-(CST)(s_j)}{\Delta s_j} =< f(t), \frac{\partial}{\partial s_j} K_s\ (t,s) > =< f(t),\ \Psi\Delta s_j(t) > .$$

Where $\Psi\Delta s_j(t) = \dfrac{1}{\Delta s_j}\left[K_s\ (t,s_1,s_2\cdots\cdots s_j+\Delta s_j\cdots s_n) - K_s\ (t,s) - \dfrac{\partial}{\partial s_j}K_s\ (t,s)\right].$

For fixed $t \in R^n$ and any fixed integer,

$$k = (k_1,k_2\cdots\cdots k_n) \in N_0^n$$

k and $D_t^k\ K_s\ (t,s)$ is analytic inside and on P¹

We have,

By Cauchy integral formula,

$$D_t^k\ \Psi\Delta s_j(t) = \frac{D_t^k}{\Delta s_j}\left[\frac{1}{2\pi i}\int_{P^1}\frac{K_s\ (t,s)dz}{z-(s_j+\Delta s_j)} - \frac{1}{2\pi i}\int_{P^1}\frac{K_s\ (t,s)}{z-s_j}dz - \frac{1}{2\pi i}\int_{P^1_1}\frac{K_s\ (t,s)}{(z-s_j)^2}dz\right.$$

$$D_t^k \Psi\Delta s_j(t) = \frac{1}{2\pi i} D_t^k \int_{P^1} K_s(t,s)\left(\frac{1}{\Delta s_j}\left(\frac{1}{z-s_j-\Delta s_j} - \frac{1}{z-s_j}\right) - \frac{1}{(z-s_j)^2}\right)dz$$

$$D_t^k \Psi\Delta s_j(t) = \frac{\Delta s_j}{2\pi i}\int_{P^1} \frac{D_t^k K_s(t,s)}{(z-s-\Delta s_j)\,(z-s_j)^2}\,dz,$$

where,

s = (s_1………s_{j-1}, z, s_{j+1}………s_n).

But for $z \in P^1$ and *t*, restricted to a compact subset of R^n,

$M(t,s) = D_t^k K_s(t,s)$ is bounded by constant M_1, therefore, we have

$$\left|D_t^k \Psi\Delta s_j(t)\right| \le \left|\Delta s_j\right| \frac{M_1}{(r_1 - r)\,(r_1)}$$

Thus as $\left|\Delta s_j\right| \to 0$, $D_t^k \Psi\Delta s_j(t)$ tends to zero. Uniformly on the compact subset's of R^n therefore, it follows that $\Psi\Delta s_j(t)$ converges in $E(R^n)$ to zero since $f \in E^1$ we concluded

$$\frac{\partial}{\partial s_j}\{CST\ f(t)\}(s) = \langle f(t), \frac{\partial}{\partial s_j} K_s(t,s) \rangle .$$

Also tends to zero,

$\therefore$ *{CST f(t)} (s)* is differentiable with respective s_j. But this is true for all *j=1,2,……n.*

Hence *{CST f(t)} (s)* is analytic on C^n and,

$$D_s^k\{CST\ f(t)\}(s) = \langle f(t), D_s^k K_s(t,s) \rangle$$

i.e. $$D_s^k\{CST\ f(t)\}(s) = \langle f(t), D_s^k K_s(t,s) \rangle$$

i.e. $$D_s^k\{CST\ f(t)\}(s) = \langle \cdot f(t), D_s^k\left(-\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot i\sin(s/b)t\right) \rangle$$

Inversion theorem

If *{CST f(t)} (s)* Canonical sine transform of *f(t)* given by,

$$\{CST\ f(t)\}(s) = -\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} i\sin(s/b)t \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

then,

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sqrt{\frac{2\pi i^3}{b}} \int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot \sin(s/b)t \cdot \{CST\ f(t)\}(s)ds$$

Proof : The canonical sine transform of f *(t)* is given by,

$$\{CST\ f(t)\}(s) = -\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} i \sin(s/b)t \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt \qquad \ldots(1.2.2)$$

$$\therefore F(s) = -\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} i \sin(s/b)t \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt.$$

Where,

$$F(s) = \{CST\ f(t)\}(s)$$

$$F(s) = -(i)\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \sin(s/b)t \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)dt$$

$$F(s)\sqrt{2\pi ib}\ \cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} = (-i)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin(s/b)t \cdot f(t)dt$$

$$F(s)i\sqrt{2\pi ib}\ \cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} = \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin(s/b)t \cdot f(t)dt.$$

$$C_1(s) = \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot f(t) \cdot \sin(s/b)t \cdot dt,$$

where,

$$F(s)\ i\sqrt{2\pi ib}\ \cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} = C_1(s)$$

and

$$g(t) = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)$$

$$C_1(s) = \int_{-\infty}^{\infty} g(t) \cdot \sin(s/b)t\ dt$$

$$C_1(s) = \int_{-\infty}^{\infty} g(t) \cdot \sin(\eta\ t)\ dt.$$

By putting $s/b = \eta$

$$\frac{1}{b}\cdot ds = d\eta \qquad \ldots(1.2.3)$$

Using inversion formula,

$$g(t) = \int_{-\infty}^{\infty} C_1(s)\cdot \sin(\eta\cdot t)\, d\eta$$

$$e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot f(t) = \int_{-\infty}^{\infty} F(s)\cdot i\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot \sin(\eta\, t)d\eta$$

$$\therefore f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty} F(s)\cdot i\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot \sin(\eta\, t)d\eta$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty} F(s)\cdot i\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot \sin(s/b)\, t\,\frac{1}{b}ds. \qquad \ldots(1.2.3)$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{2\pi i^3}{b}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot \sin(s/b)\, t\, F(s)\cdot ds.$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{2\pi i^3}{b}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot \sin(s/b)\, t\,\{CST\ f(t)\}\,(s)ds$$

Scaling property of canonical sine transform

If {*CST f(t)} (s)* denotes generalized canonical sine transform of *f(t)*, then,

$$\{CST\ \ f(t)\}(s) = \frac{1}{k}e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\frac{d}{bk}s^2}\left[CST\ \ f(t)\,e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2}\right](s)$$

Proof: $\{CST\ f(kt)\}(s) = (-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot \sin(s/b)t\cdot f(kt)dt$

Put, kt = T, dt = $\frac{1}{k}$ dT

$$\{CST\ f(kt)\}(s) = (-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)\frac{T^2}{k^2}}\cdot \sin(s/b)\left(\frac{T}{k}\right)\cdot F(T)\frac{dT}{k}$$

$$= (-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)\frac{T^2}{k^2}}\cdot \sin\left(\frac{s}{bk}\right)T\,F(T)\frac{dT}{k}$$

$$= (-i)\frac{1}{k}\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{bk}\right)t^2} \cdot e^{-\frac{i}{2}\left(\frac{a}{bk}\right)t^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)\frac{t^2}{k^2}} \sin\left(\frac{s}{bk}\right) t\, f(t)\,dt$$

$$= (-i)\frac{1}{k}\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{d}{bk}\right)s^2} \cdot e^{-\frac{i}{2}\left(\frac{d}{bk}\right)s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{bk}\right)t^2} \sin\left(\frac{s}{bk}\right) t\, f(t) \cdot e^{-\frac{i}{2}\left(\frac{a}{bk}\right)t^2}$$
$$e^{\frac{i}{2}\left(\frac{a}{b}\right)\frac{t^2}{k^2}} \,].dt$$

$$= \frac{1}{k}(-i)\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{bk}\right)s^2} \cdot e^{-\frac{i}{2}\left(\frac{d}{bk}\right)s^2} \cdot e^{\frac{i}{2}\frac{d}{b}s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{bk}\right)t^2} \sin\left(\frac{s}{bk}t\right)$$
$$\left[f(t) \cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2} \right] dt$$

$$= \frac{1}{k} \cdot e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\frac{d}{bk}s^2} \int_{-\infty}^{\infty} \left[e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2} \cdot f(t) \right] K_s(t,s)\,dt$$

$$\{CST\ f(kt)\}(s) = \frac{1}{k} \cdot e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\frac{d}{bk}s^2} \int_{-\infty}^{\infty} \left[e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2} \cdot f(t) \right] K_s(t,s)\,dt$$

$$\{CST\ f(kt)\}(s) = \frac{1}{k} \cdot e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\frac{d}{bk}s^2} \left[CST\left\{ f(t) \cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2} \right\} \right](s)$$

CONCLUSION

In this paper, brief introduction of the generalized canonical sine transform is given and its analyticity theorem, Inverse theorem is proved. Scaling property of canonical sine transform is also obtained which will be useful in solving differential equations occurring in signal processing and many other branches of engineering.

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