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FUZZY DYNAMIC EQUATIONS ON TIME SCALES UNDER SECOND TYPE HUKUHARA DELTA DERIVATIVE

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ABSTRACT

In this paper, we introduce and study the properties of second type Hukuhara delta derivative denoted by (Δ_{SH}–derivative) for fuzzy set-valued functions on time scales whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n. We establish the existence and uniqueness criteria for fuzzy dynamic equations on time scales using Banach contraction principle. For an application, we consider the radioactive decay problem and illustrate the advantage of (Δ_{SH}–derivative).

Key words: Fuzzy set-valued mapping, H-derivative, Δ_{SH} – Derivative, Time scales.

INTRODUCTION

Hukuhara derivative introduced by Hukuhara[1] is the starting point for the study of Set Differential Equations (SDEs) and later for Fuzzy Differential Equations (FDEs). The Hukuhara differentiability (H-differentiability) of fuzzy mapping defined by Puri and Ralescu[2] was the first approach for modeling the uncertainity of the dynamical systems. FDEs are appropriate in modeling of many real-world phenomena, where some uncertainities arise due to inexactness and impreciseness. FDEs play an important role both in theory and applications[3-11]. In[12-15] the authors studied the existence and uniqueness of the solutions of Fuzzy differential equations using H-derivative. But this approach has the disadvantage that it leads to solutions which have an increasing length of their support. Consequently, this approach cannot reflect the rich behavior of FDEs. Hence, the generalization of the concept of H-differentiability can be of great help in the dynamic study of FDEs. To overcome this situation, the authors in[16,17] introduced the concept of strongly

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generalized differentiability to study the Fuzzy- number-valued functions. Using this generalized differentiability concept, the authors in[18] introduced the concept of lateral H-derivative, which leads to different solutions for FDEs[19]. In[20,21], the authors used the concept of generalized Hukuhara difference and studied the interval-valued functions and interval differential equations[22,23]. Recently, in our paper[24], we introduced the concept of Δ_g–derivative and Δ_g–integral using Hukuhara difference and studied various properties of fuzzy set-valued functions on time scales.

A dynamic model describes the behavior of a system by differential or difference equations. Hilger[25] introduced and developed the theory of time scales that can unify the study of discrete and continuous dynamic systems. For calculus on time scales we refer[26-31]. To analyze a real world phenomenon, it is necessary to handle number of uncertain factors. In that case, the theory of fuzzy sets is one of the best approaches, which lead us to fuzzy dynamical models. Hukuhara derivative of multivalued functions on time scales was introduced in[32]. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales using generalized Hukuhara difference was studied in[33].

In this paper, we focus our attention on fuzzy dynamic equations on time scales using Δ_{SH}-derivative. We present some sufficient conditions under which the fuzzy dynamic equations with Δ_{SH}-derivative on time scales have solutions, which have decreasing level of uncertainity. The paper is organized as follows. In section 2, some basic definitions and results related to fuzzy and time scale calculus are presented. In section 3, we introduce and study the new class of derivative called second type Hukuhara delta derivative (Δ_{SH}-derivative) for fuzzy set-valued mappings on time scales. In section 4, we establish the sufficient condition for the existence and uniqueness of the solution of fuzzy dynamical equation on time scales and the results are illustrated with the real world application of radioactive decay problem.

Preliminaries

In this section, we present some definitions, properties and results on fuzzy and time scale calculus, which are useful for later discussion. Let $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of R^n. Define the addition and scalar multiplication ($\bullet$) in $P_k(R^n)$ as usual. Moreover, if $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then

$$\alpha \bullet (A + B) = \alpha \bullet A + \alpha \bullet B, \alpha \bullet (\beta A) = (\alpha \beta) \bullet A, 1 \bullet A = A$$

and if $\alpha, \beta \geq 0$ then $(\alpha + \beta) \bullet A = \alpha \bullet A + \beta \bullet A$. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

where $\|.\|$ denotes the Euclidean norm in R^n. Denote $E^n = \{u : R^n \to [0,1]\}$, and u satisfies (i)-(iv) below where –

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) the closure of $\{x \in R^n/u(x) > 0\}$, denoted by $[u]^0$ is compact.

For $0 \le \alpha \le 1$, denote $[u]^\alpha = \{x \in R^n/u(x) \ge \alpha\}$, then from (i)-(iv) it follows that the α-level set $[u]^\alpha \in p_k(R^n)\ \forall\ 0 \le \alpha \le 1$.

For any $u, v \in E^n$, define $D(u,v) = \sup_{0\le\alpha\le1} d_H([u]^\alpha, [v]^\alpha)$.

Lemma 2.1.[13] For any $A, B, C, D \in E^n$ and $\lambda \in R$,

(i) (E^n, D) is a complete metric space

(ii) $D(A+C, B+C) = D(A,B)$,

(iii) $D(\lambda \bullet A, \lambda \bullet B = |\lambda|\, D(A,B)$.

For any $A, B \in E^n$, if there exists a $C \in E^n$ such that $A = B + C$, then we call C the Hukuhara difference of A and B denoted by $A \Theta B$ [10]. It is known that $A \Theta B$ exists in the case $\text{diam}(A) \ge \text{diam}(B)$. Also one can verify the following properties for $A, B, C, D \in E^n$.

Lemma 2.2.[17] Let $A, B \in E^n$, then

(i) If $A \Theta B$ exists, then $A \Theta A = \{0\}$, (ii) $(A+B) \Theta B = A$,

(ii) If $A \Theta B$, $A \Theta C$ exist, then $D(A \Theta B, A \Theta C) = D(B,C)$,

(iii) $A \Theta B = \{0\} \Leftrightarrow D(A,B) = 0$,

(iv) If $A \Theta B$, $C \Theta D$ exist, then $D(A \Theta B, C \Theta D) = D(A + D, B + C)$.

Let $T = [a, b] \in R$ be a compact interval.

Definition 2.1.[4] We say that a mapping $F : T \to E^n$ is strongly measurable if $\forall$ $\alpha \in [0,1]$ the set valued mapping $F_\alpha : T \to P_k(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is (Lebesgue) measurable, when $P_k(R^n)$ is endowed with the topology generated by the Hausdorff metric d_H.

Definition 2.2.[4] Let $F : T \to E^n$. The integral of F over T, denoted by –

$\int_T F(t)dt$ or $\int_a^b F(t)dt$, is defined levelwise by the equation

$$[\int_T F(t)]^\alpha = \int_T F_\alpha(t)dt = \{\int_T F(t)dt / f : T \to R^n\}$$

where f is a measurable selection for F_α $\forall$ $0 < \alpha \leq 1$.

Definition 2.3.[13] A mapping $F : T \to E^n$ is said to be Hukuhara differentiable at $t_0 \in T$ if there exists a $F'(t_0) \in E^n$ such that $F(t_0 + h) \Theta F(t_0), F(t_0) \Theta F(t_0 - h)$ exists $\forall$ h > 0 sufficiently small such that the limits exist in the topology of E^n and equal to $F'(t_0)$.

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \Theta F(t_0)}{h}, \lim_{h \to 0^+} \frac{F(t_0) \Theta F(t_0 - h)}{h}$$

The element $F'(t_0)$ is called the Hukuhara derivative of F at t_0 taken in the metric space (E^n, D). At the end points of T we consider only the one-sided derivatives. For the properties on Hukuhara derivative, we refer to[13].

Now, we will present some basic definitions and results related to time scales calculus.

Definition 2.4.[28] Let T be a time scale. The forward jump operator $\sigma : T \to T$, the backward jump operator and the graininess $\mu : T \to R^+$ are defined by $\sigma(t) = \inf\{s \in T : s > t\}$, $\rho(t) = \inf\{s \in T : s < t\}$, $\mu(t) = \sigma(t) - t$, for $t \in T$, respectively.

If $\sigma(t) = t$, t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered). If T has a left- scattered maximum m, then $T^k = T - \{m\}$. Otherwise $T^k = T$. If $f : T \to R$ is a function, then we define the function $f^\sigma : T \to R$ by $f^\sigma(t) = f(\sigma(t)) \, \forall \; t \in T$.

Definition 2.5.[28] Assume that $f : T \to R$ is a function and let $t \in T^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap T$) for some $\delta > 0$ such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon |\sigma(t) - s|, \forall \; s \in U$$

In this case, $f^\Delta(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta (or Hilger) differentiable on T if $f^\Delta(t)$ exists $\forall \; t \in T^k$. The function $f^\Delta : T^k \to R$ is then called the delta derivative of f on T^k.

Definition 2.6.[28] A function $f : T \to R$ is called regulated provided its right-sided limits exist(finite) at all right dense points in T and its left-sided limits exist(finite) at all left-dense points in T and F is said to be rd-continuous if it is continuous at all right-dense points in T and its left-sided limits exists(finite) at all left-dense points in T.

Definition 2.7.[28] Let $f : T \to R$ be a mapping. The mapping $g : T \to R$ is called an anti-derivative of f on T if it is differentiable on T and $g^\Delta(t) = f(t)$ for $t \in T$.

Lemma 2.3.[28] Assume $f : T \to R$

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right scattered, then f is delta differentiable and $f^\Delta(t) = \dfrac{f(\sigma(t) - f(t))}{\mu(t)}$.

(iii) If f is delta differentiable at t, then

(a) $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$,

(b) $f(\sigma(t)) = f(t) - \mu(\rho(t)) f^\Delta(\rho\rho(t)$

Differentiability and integrability of fuzzy set valued functions on time scales

In this section we define and study the properties of second type Hukuhara delta derivative (Δ_{SH}–derivative) for fuzzy set-valued functions on time scales. To facilitate the discussion below, we introduce some notation: For $t \in T$ the neighbourhood t of T is denoted by $U_T = U_T(t,\delta)$, $(U_T = (t-\delta, t+\delta) \cap T)$ for some $\delta > 0$. In the present section we work in (E^n, D).

Definition 3.1.[24] A fuzzy set-valued function $F : T \to E^n$ has a T-limit $A \in E^n$ at $t_0 \in T$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $D(F(t)\,\Theta\, A, \{0\} \leq \varepsilon)$ $\forall$ $t \varepsilon\, U_T$. If F has a T-limit $A \in E^n$ at $t_0 \in T$, then it is unique and is denoted by $T - \lim\limits_{t \to t_0} F(t)$. F is continuous at $t_0 \in T$, if $T - \lim\limits_{t \to t_0} F(t)$ exists and $T - \lim\limits_{t \to t_0} F(t) = F(t_0)$.

Definition 3.2. Let $F : T \to E^n$ be a fuzzy set-valued function and $t \in T^k$. Then F is said to be second type Hukuhara delta differentiable (Δ_{SH}–differentiable) at $t \in T^k$, if there exists $\Delta_{SH}F(t) \in E^n$ with the property that given any $\varepsilon > 0$, there is a neighbourhood U_T of t for some $\delta > 0$ such that –

$$D[F(\sigma(t)\,\Theta\, Ft + h), \Delta_{sH}F(t)(-(h-\mu(t))))] \leq \varepsilon(-(h-\mu(t))) \qquad \ldots(3.1)$$

$$D[F(t-h)\,\Theta\, F(\sigma(t)), \Delta_{sH}F(t)(-(h+\mu(t))))] \leq \varepsilon(-(h+\mu(t))) \qquad \ldots(3.2)$$

$\forall$ $t-h$, $t+h \in U_T$ with $o < h < \delta$. We call $\Delta_{SH}F(t)$ be the second type Hukuhara delta derivative (Δ_{SH}–derivative) of F at t. We say F is Δ_{SH}–differentiable on T^k, if its Δ_{SH} derivative exists at each $t \in T^k$. The fuzzy set valued function $\Delta_{SH}F : T \to E^n$ is then called the Δ_{SH}–derivative of F on T^k.

Remark 3.1. Let $F : T \to E^n$ be a fuzzy set-valued function and $t \in T^k$. Then Definition 3.2 can be equivalently written as

(i) For $0 > h < \delta$ sufficiently small, there exists the H-difference $F(\sigma(t)\,\Theta\, F(t+h))$ and the limit exists in the metric D.

$$\lim_{h \to 0^+} \left(\frac{-1}{h-\mu(t)} \right) \bullet (F(\sigma(t))\,\Theta\, F(t+h)) = \Delta_{SH}F(t) \qquad \ldots(3.3)$$

(ii) For $0 > h < \delta$ sufficiently small, there exists the H-difference $F(t - h)\ \Theta\ F(\sigma(t))$ and the limit exists in the metric D.

$$\lim_{h \to 0^+} \left(\frac{-1}{h + \mu(t)} \right) \bullet (F(t - h)\ \Theta\ F(\sigma(\sigma(t) = \Delta_{SH} F(t) \qquad \ldots(3.4)$$

Remark 3.2. The Δ_{SH} –derivative defined in Definition 3.2 coincides with the equations (2) and (4) in definition 11 of.[24] If T = R, the Δ_{SH} –differentiability coincides with the Definition 3(ii) of[18] and also coincides with the strongly generalized differentiability given in Definition 5(ii) of[16].

Theorem 3.1. Let $F : T \to E^n$ be fuzzy set-valued function and let $t \in T^k$. Then

(i) If F is Δ_{SH} –differentiable at $t \in T^k$ then it is continuous at t.

(ii) If F is continuous at t and t is right-scattered then F is Δ_{SH} –differentiable at t and $\Delta_{SH} F(t) = \frac{1}{\mu(t)} \bullet (F(\sigma(t))\ \Theta\ F(t)) = \frac{1}{\mu(t)} \bullet (F(t)\ \Theta\ F(\sigma(t)))$.

(iii) If t is right-dense, then F is Δ_{SH} –differentiable at $t \in T^k$. Then $\frac{\lim}{h \to 0^+} \frac{-1}{h} \bullet (F(t)\ \Theta\ F(t + h)) = \frac{\lim}{h \to 0^+} \frac{-1}{h} \bullet (F(t - h)\ \Theta\ F(t)) = F'(t)$.

(iv) If F is Δ_{SH} –differentiable at $t \in T^k$. Then $F(\sigma(t)) = F(t)\ \Theta\ (-1)\ \mu(t) \Delta_{SH} F(t)$ or $F(t) = F(\sigma(t)) + (-1)\ \mu(t) \Delta_{SH} F(t)$.

Proof: The proof is similar to the proof of Theorem 1 in[24].

Example 3.1. Consider $F : T \to E^1$ defined by $F(t) = (-t) \bullet u, \forall\ t \in T,$ where u = (2, 3 ,4) is a triangular fuzzy number. If $T = R$, then $\sigma(t) = t$ and $\mu(t) = 0$. From[17], the H-differences $F(t + h)\ \Theta\ F(t)$, $F(t)\ \Theta\ F(t - h))$ cannot exists and hence $F'(t)$ does not exist. From Theorem1 (iii),

$$F'(t) = \frac{\lim}{h \to 0} \frac{-1}{h} \bullet (F(t)\ \Theta\ F(t + h))$$

$$= \frac{\lim}{h \to 0} \frac{-1}{h} \bullet ((-t) \bullet (2,3,4)\ \Theta\ (-(t + h)) \bullet (2,3,4))$$

$$= (-1) \bullet (2,3,4) = (-4,-3,-2).$$

In a similar way, we can prove $\lim_{h \to 0} \frac{-1}{h} \bullet (F(t-h)\,\Theta\, F(t)) = (-4,-3,-2).$

If $T = Z$, then $\sigma(t) = t+1$ and $\mu(t) = 1$. Hence every point in Z is right-scattered. Then from Theorem 1 (ii) yields that $F : Z \to E^1$ is Δ_{SH}–differentiable and

$$\Delta_{SH}F(t) = \frac{1}{\mu(t)} \bullet (F(\sigma(t))\,\Theta\, F(t))$$

$$= F(t+1)\,\Theta\, F(t) = -(t+1)\bullet(2,3,4)\,\Theta\,(-t)\bullet(2,3,4)$$

$$= (-1)\bullet(2,3,4) = (-4,-3,-2).$$

Remark 3.3. Let $F : T \to E^n$ be fuzzy set-valued function. Then if F is Δ_{SH} –differentiable at $t \in T^k$, then there exists $\delta > 0$ such that for $\alpha \in [0,1]$ and for $0 < h < \delta$,

$$\text{diam}[F(t+h)]^\alpha \le \text{diam}[F(\sigma(t))]^\alpha \le \text{diam}[F(t-h)]^\alpha$$

Hence if F is Δ_{SH}–differentiable then $\text{diam}[F(t)]^\alpha$ is nonincreasing on T^k and hence the solution has decreasing length of support i.e. uncertainity decreases as time increases which is the main advantage of Δ_{SH}–derivative.

Theorem 3.2. Let $F, G : T \to E^n$ are Δ_{SH}–differentiable at $t \in T^k$. Then,

(i) The sum $F + G : T \to E^n$ is Δ_{SH}–differentiable at $t \in T^k$ with

$\Delta_{SH}(F+G)(t) = \Delta_{SH}F(t) + \Delta_{SH}G(t);$

(ii) The H-difference $F\,\Theta\, G : T \to E^n$ is Δ_{SH}–differentiable at $t \in T^k$ with

$\Delta_{SH}(F\,\Theta\, G)(t) = \Delta_{SH}F(t)\,\Theta\,\Delta_{SH}G(t);$

(iii) For any constant λ, $\lambda F : T \to E^n$ is Δ_{SH}–differentiable at $t \in T^k$ with

$\Delta_{SH}(\lambda \bullet F)(t) = \lambda \bullet \Delta_{SH}F(t);$

(iv) The product $FG : T \to E^n$ is Δ_{SH}–differentiable at $t \in T^k$ with

$$\Delta_{SH}(FG)(t) = G(t)\Delta_{SH}F(t) + F(\sigma(t))\Delta_{SH}G(t);$$
$$= G(\sigma(t))\Delta_{SH}F(t) + F(t)\Delta_{SH}G(t).$$

Proof: The proof is similar to the proof of Theorem 2 in[24].

Lemma 3.1. Let $F: T \to E^n$ be fuzzy set-valued function and denote $[F(t)]^\alpha = F_\alpha(t)$, for each $\alpha \in [0,1]$. If F is Δ_{SH}–differentiable at $t \in T^k$, then F_α is also Δ_{SH}–differentiable on T^k and $\Delta_{SH}[F(t)]^\alpha = \Delta_{SH} F_\alpha(t),\ \forall\ t \in T^k$.

Proof: If F is Δ_{SH} –differentiable at $t \in [a,b)_T$ then for $0 < h < \delta$ and for any $\alpha \in [0,1]$, we ge $[(F(\sigma(t))\,\Theta\, F(t+h))]^\alpha = [(F_\alpha(\sigma(t))\,\Theta\, F_\alpha(t+h))]$ and dividing by $(-(h-\mu(t))$ and let $h \to 0^+$ we have

$$\lim_{h\to 0^+}\left(\frac{-1}{h-\mu(t)}\right)\bullet[(F_\alpha(\sigma(t))\,\Theta\, F_\alpha(t+h))] = \Delta_{SH} F_\alpha(t).$$

Similarly, $\lim_{h\to 0^+}\left(\frac{-1}{h+\mu(t)}\right)\bullet(F_\alpha(t-h)\,\Theta\, F_\alpha(\sigma(t))) = \Delta_{SH} F_\alpha(t).$

Definition 3.3.[24] Let $I \subset T$. A function $f: I \to R$ is called a Δ-measurable sector of the fuzzy set valued function $F: I \to E^n$ if $f(t) \in F(t)\ \forall\ t \in I$ and f is said to be regulated Δ-measurable sector if it is regulated. Similarly, f is said to be rd-continuous Δ-measurable sector if it is rd-continuous.

Definition 3.4.[24] A fuzzy set-valued function $F: T \to E^n$ is said to be Δ_{SH}–integrable on $I \subset T$ if F has a rd-continuous Δ-measurable sector on I. In this case, we define the Δ_{SH}–integral of F on I, denoted by $\int_I F(s)\,\Delta s$, and defined levelwise by the equation.

$$\left[\int_I F(s)\Delta s\right]^\alpha = \int_I F_\alpha(s)\Delta s = \left\{\int_I f(s)\Delta s : f \in S_{F_\alpha}(I)\right\}$$

where $S_{F_\alpha}(I)$, the set of all Δ_{SH}–integrable sectors of F_α on I.

Lemma 3.2.[24] Let $F, G: [t_0, \tau]_T \to E^n$ are Δ_{SH}–integrable and have rd-continuous Δ –measurable sectors, then we have

(i) $\int_{t_0}^{\tau}[F(s)+G(s)]\Delta s = \int_{t_0}^{\tau} F(s)\Delta s + \int_{t_0}^{\tau} G(s)\Delta s;$

(ii) $\int_{t_0}^{\tau} \lambda \bullet F(s)\Delta s = \lambda \bullet \int_{t_0}^{\tau} F(s)\Delta s, \lambda \in R_+$

(iii) $\int_{t_0}^{\tau} [F(s)\Delta F = \int_{t_0}^{t} F(s)\Delta s + \int_{t}^{\tau} F(s)\Delta s$

(iv) $D(\int_{t_0}^{\tau} F(s)\Delta s, \theta) \le \int_{t_0}^{t} D(F(s), \theta)\Delta s$

(v) $D(\int_{t_0}^{\tau} F(s)\Delta s, \int_{t_0}^{\tau} G(s)\Delta s) \le \int_{t_0}^{t} D(F(s), G(s))\Delta s;$

Theorem 3.4. Let $F:[t_0,\tau]_T \to E^n$ be rd-continuous. If F is Δ_{SH}–integrable from t_0 to τ then the fuzzy set-valued function $G:[t_0,\tau]_T \to E^n$ given by $G(t) = \int_{t_0}^{t} F(s)\Delta s, t \in [t_0, t]_T$ is continuous on $t \in [t_0, t]_T$. Further for $t \in [t_0, t)_T$ and let F be arbitrary at t, if t is right-scattered, and let F be continuous at t if t is right-dense. Then G is Δ_{SH}–integrable at t and $\Delta_{SH} G(t) = F(t) \forall \ t \in [t_0, t)_T$.

Proof: Let $t \in [t_0, t)_T$ be right-scattered. Since $G:[t_0,\tau]_T \to E^n$ is continuous from Theorem1 (ii), it follows that G is Δ_{SH}–differentiable at t and hence we have

$$\Delta_{SH} G(t) = \frac{1}{\mu(t)} \bullet (G(\sigma G(\sigma(t)\, t))) = \frac{1}{\mu(t)} \bullet (\int_{t_0}^{\sigma(t)} F(s)\, \Delta s\, \Theta \int_{t_0}^{t} F(s)\Delta s)$$

$$= \frac{1}{\mu(t)} \bullet (\int_{t}^{\sigma(t)} F(s)\Delta s) = F(t).$$

If t is right-dense and F is continuous at t, then from Theorem1 (iii), it follows that

$$(G(\sigma G(\sigma(t)\, t + h))) = \int_{t_0}^{\sigma(t)} F(s)\Delta(\Theta \int_{t_0}^{t+h} F(s)\Delta s = \int_{t+h}^{\sigma(t)} F(s)\Delta s$$

$$(G(t-h)\, \Theta G\, (\sigma\, (t)) = \int_{t_0}^{t-h} F(s)\Delta(\Theta \int_{t_0}^{\sigma(t)} F(s)\Delta s = \int_{\sigma(t)}^{t-h} F(s)\Delta s$$

Let $\varepsilon > 0$, by the continuity of F we have –

$$D\left(\frac{-1}{h-\mu(t)} \bullet (G(\sigma G(\sigma(t)(t+h)), F(t)\right) = \frac{-1}{h-\mu(t)} \bullet D\left(\int_{t+h}^{\sigma(t)} F(s)\Delta(, \int_{t+h}^{\sigma(t)} F(t)\Delta s\right)$$

$$= \frac{-1}{h-\mu(t)} \bullet \int_{t+h}^{\sigma(t)} D\big(F(s), F(t)\big)\Delta s < \varepsilon,$$

for $0 < h < \delta$ sufficiently small. Hence $\Delta_{SH}G(t) = F(t)\ \forall\ t \in [t_0, t)_T$.

Remark 3.4. Let $F : [t_0, \tau]_T \to E^n$ be rd-continuous. If F is Δ_{SH} –integrable on $[t_0, t)_T$, then

$$F(\tau(= F(t_0)\Theta\,(-1)\int_{t_0}^{\tau} \Delta_{SH}F(s)\Delta\, s.$$

Fuzzy dynamic equations on time scales

In this section we consider a fuzzy initial value problem (IVP) on time scales

$$y^{\Delta} = F(t, y),\ y(t_0) = y_0 \qquad \ldots(4.1)$$

Where the derivative Δ denotes the Δ_{SH} –derivative and $F : T^k \times E^n \to E^n$ is rd-continuous, $t_0 \in$ T and $y_0 \in E^n$. Let $C_{rd}([a, \sigma(b))]_T, E^n)$ be the set of all rd-continuous fuzzy functions from $[a, \sigma(b))]_T \to E^n$. The solution $y : T^k \to E^n$ is unique if $\sup_{t \in T^k} D(x(t), y(t)) = 0$, $\forall\ t \in [a, \sigma(b))]_T$. If x(t) is an antiderivative of F(t, y(t)) on T^k and which is a Δ_{SH} – differentiable solution to (4.1).

Lemma 4.1. A fuzzy function $y \in C_{rd}([a, \sigma(b))]_T, E^n)$ is called a Δ_{SH}– differentiable solution to the IVP (4.1) if and only if it satisfies the integral equation

$$y_0 = y(t) + (-1)\int_{t_0}^{t} F(s, y(s))\Delta\, s,\ \forall\ t \in [a, \sigma(b))]_T \qquad \ldots(4.2)$$

$$y(t) = y_0\Theta(-1)\int_{t_0}^{t} F(s, y(s))\Delta\, s,\ \ t \in [a, \sigma(b))]_T \qquad \ldots(4.3)$$

The following definition and remark are simple extension of Definition 8.14. of[28].

Definition 4.1. A fuzzy mapping $F: T^k \times E^n \to E^n$ is said to be

(i) rd-continuous, if g defined by $g(t) = F(t, y(t))$ is rd-continuous for any continuous function $y: T^k \to E^n$;

(ii) Bounded on a set $S \subset T^k \times E^n$, if there exist a constant $M > 0$ such that $D(F(t, y), \hat{0}) \le M \ \forall \ (t, y) \in S$;

(iii) Lipschitz continuous on a set $S \subset T^k \times E^n$, if there exist a constant $L > 0$ such that $D(F(t, y_1), F(t, y_2)) \le LD(y_1, y_2),\ (t, y_1), (t, y_2) \in S$;

(iv) Regressive at $t \in T^k$, if the mapping $id + \mu(t)F(t,.): E^n \to E^n$ is invertible (where id is the identity fuction), and F is regressive on T^k, if F is regressive at each $t \in T^k$.

Remark 4.1: A Lipschitz function $F: T^k \times E^n \to E^n$ is regressive on T^k, provided the Lipschitz constant L satisfies $L\,\mu(t) < 1 \ \forall \ t \in T^k$.

Theorem 4.1: (Local Existence and Uniqueness Theorem**)** Let $F: [a,b]_T \times E^n \to E^n$ be rd-continuous and Lipschitz continuous with constant L > 0. Then

(i) If t_0 is right-scattered then there exists a unique Δ_{SH} – differentiable solution to (4.1) on the interval $[a, \sigma(b))]_T$.

(ii) If t_0 is left-scattered then there exists a unique Δ_{SH} – differentiable solution to (4.1) on the interval $[a, \sigma(b))]_T$ provided Fis regressive.

Proof Let $C = C_{rd}([a, \sigma(b))]_T, E^n)$ be the set of all rd-continuous fuzzy functions from $[a, \sigma(b)]_T \to E^n$. Define the operator $A_1: C \to C$ by –

$$[A_1 y](t) = y_0 \Theta(-1) \int_{t_0}^{t} F(s, y(s)) \Delta s, \quad t \in [a, \sigma(b))]_T$$

From Lemma 7 $A_1 y \in C$. Considering the metric D_ρ on C, defined by –

$$D_\rho(x, y) = \sup_{s \in [a, \sigma(b)]_T} \{D(x\,(s, y(s)) e_{-\rho}(s, 0))\}, \quad x, y \in C,$$

Where $\rho > 0$ large enough such that $\frac{1-e_{-\rho}(T,0)}{\rho} < 1$

Clearly, $(C_{rd}([a,\sigma(b))]_T, E^n), D_\rho)$ is a complete metric space. Furthermore, by Lemma 6 and by the Lipschitz continuity of F we have –

$$D_\rho(A_1x, A_1y) = \sup_{t\in[a,\sigma(b)]_T} \{D([A_1x](t),[A_1y](t))e_{-\rho}(t,0)\}$$

$$= \sup_{t\in[a,\sigma(b)]_T} \left\{ D\left(y_0\Theta(-1)\int_0^t F(s,x(s))\Delta(s \quad y_0\Theta(-1)\int_0^t F(s,y(s))\Delta s \right) e_{-\rho}(t,0) \right\}$$

$$= \sup_{s\in[a,\sigma(b)]_T} \left\{ D\left(\int_0^t F(s,x(s))\Delta(s \int_0^t F(s,y(s))\Delta s \right) e_{-\rho}(t,0) \right\}$$

$$\leq \sup_{t\in[a,\sigma(b)]_T} \left\{ \int_0^t D\big(F(s,x(s)), \quad F(s,y(s))\big)\Delta s \;\; e_{-\rho}(t,0) \right\}$$

$$\leq \sup_{t\in[a,\sigma(b)]_T} \left\{ \int_0^t D(x(s),y(s))\Delta s \, e_{-\rho}(t,0) \right\}$$

$$= \sup_{t\in[a,\sigma(b)]_T} \left\{ \int_0^t D(x(s),y(s)) \;\; e_{-\rho}(s,0) \;\; e_\rho(s,0) \;\; \Delta s \;\; e_{-\rho}(t,0) \right\}$$

$$\leq \sup_{t\in[a,\sigma(b)]_T} \left\{ D_\rho(x,y)\int_0^t e_\rho(s,0) \;\; \Delta s \, e_{-\rho}(t,0) \right\}$$

$$\leq D_\rho(x,y) \sup_{t\in[a,\sigma(b)]_T} \left\{ \left(\frac{e_\rho(t,0)-1}{\rho} \right) e_{-\rho}(t,0) \right\}$$

$$= D_\rho(x,y) \sup_{t\in[a,\sigma(b)]_T} \left\{ \left(\frac{1-e_{-\rho}(t,0)}{\rho} \right) \right\} = \left(\frac{1-e_{-\rho}(T,0)}{\rho} \right) D_\rho(x,y)$$

Therefore $\quad D_\rho(A_1x, A_1y) \leq \left(\frac{1-e_{-\rho}(T,0)}{\rho} \right) D_\rho(x,y), \quad \forall x,y \in C_{rd}([a,\sigma(b)])_T$

So, A_1 is a contraction mapping. Hence by Banach contraction mapping theorem A_1 has unique Δ_{SH} – differentiable solution $\bar{\lambda}$ to the IVP (4.1).

(i) If t_0 is right-scattered then is uniquely determined.

$$\vec{\lambda}(\sigma(t_0)) = \vec{\lambda}(t_0) + \mu(t)\vec{\lambda}^{\Delta}(t_0) = y_0 + \mu(t)F(t_0, y_0),$$

Hence (4.1) has unique Δ_{SH} – differentiable solution on $[a, \sigma(b)]_T$, when t_0 is right-scattered.

(ii) If t_0 is left-scattered then

$$\begin{aligned} y_0 &= \vec{\lambda}(t_0) = \vec{\lambda}(\rho(t_0)) + \mu(\rho(t_0))\, F(\rho(t_0), \vec{\lambda}^{\Delta}(\rho(t_0))) \\ &= (id + \mu F(\rho(t_0), \ .))\vec{\lambda}(\rho(t_0)). \end{aligned}$$

Since Fis regressive, $\vec{\lambda}(\rho(t_0)) = (id + \mu F(\rho(t_0), .))^{-1} y_0$ is uniquely determined. Hence (4.1) has unique solution on $[a, \sigma(b)]_T$, when t_0 is left-scattered.

The following example illustrates the importance of Δ_{SH}–derivative.

Example 4.1. Let us consider the radioactive decay problem. As radioactive decay is entirely a random process, it is impossible to predict which atoms of the radioactive substance is undergoing radioactive decay at a moment of time which can be modeled by the fuzzy dynamic equation

$$y^{\Delta}(t) = -k \bullet y(t), \ y(0) = y_0 \qquad \ldots(4.4)$$

Where y(t) denotes the number of radioactive nuclei present at time $t > 0$, k is the proportionality constant for the radioactive substance, Δ denotes the Δ_{SH} – derivative and $y_0 \in E^n$. In this problem, the uncertainity is introduced in y_0 due to uncertain information on the initial number of radioactive nuclei present in the substance. Let $y_0 = (1, 2, 3)$, a triangular fuzzy number, $k = 1$ and time scale $T = R$, then the corresponding solution of (4.4) is –

$$y(t) = e^{-t} \bullet (1, 2, 3) = (e^{-t}, 2e^{-t}, 3e^{-t}) \qquad \ldots(4.5)$$

As $t \to \propto$, $e^{-t} \to 0$, i.e. uncertainity decreases with time t and disappears asymptotically, which is represented as in Fig. 1. Since radioactivity of a material always decreases with time, this Δ_{SH} – differentiable solution is the appropriate solution for the modeling of radioactive decay problem under the presence of uncertainity.

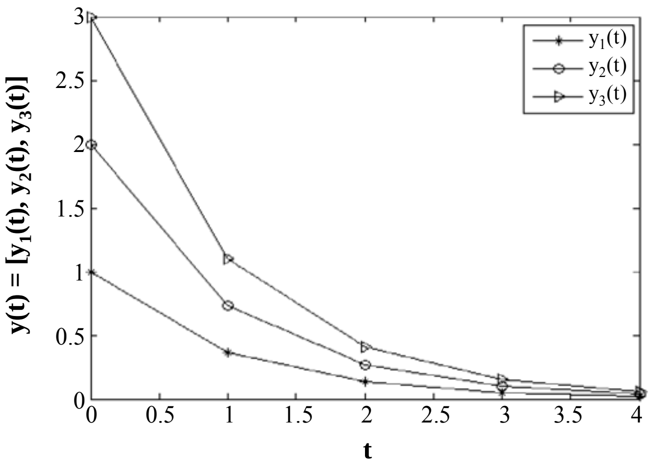

Fig. 1: Solution of the fuzzy dynamical equation (4.4) using Δ_{SH} –differentiability

Hence, under the assumptions of Theorem 4.1., it is easy to find out the solution for (4.5) as we move forward with time but Lipschitz continuity on F alone is not sufficient as we move backward in time which can be seen from the following example.

Example 4.2. Consider the fuzzy dynamic equation

$$y^{\Delta}(t) = -y(t), y(0) = y_0 \quad \ldots(4.6)$$

With the time scale T = Z. Hence (4.6) becomes $\Delta y(t) = y(t), \quad y(0) = y_0$, where Δ is the forward difference operator. Hence F(t, y) = – y Clearly,

$$D(F(t,y_1), F(t,y_2)) \leq D(y_1, y_2),$$

and hence F is Lipschitz continuous with L = 1. Moreover, we have $y(t) = 0, \forall t \in N$. However the solution does not exist at –t for $t \in N$. Since μ(t) = 1 for the time scale T = Z and hence F is not regressive from Remark 5. Hence a solution could exist for all times but may not be unique, if regressivity is not satisfied.

CONCLUSION

In this paper, we deal with fuzzy dynamic equations on time scales with second type Hukuhara delta derivative (Δ_{SH} – derivative). These dynamic equations are appropriate tool for the engineers in modeling the dynamical systems under the presence of uncertainity caused by the lack of exact information about the parameters of dynamical systems. The radioactive decay problem is considered with uncertain information on initial condition and illustrated existence and uniqueness result when time scale T = R and T = Z. The advantage of this Δ_{SH} – derivative is that the solutions of the corresponding fuzzy dynamic equations have decreasing length of uncertainity as the time increases.

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