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Electromagnetic Field, Spin, and Gravitation as Characteristics of a Charged Quantum Particle Wave Function

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Abstract

In this paper, a quantum particle is conceived as a packet of waves with invariant time-dependent phases for any change of coordinates. In this framework, we obtain the relativistic kinematics and dynamics, and the spin as a characteristic of the particle dynamics. When a field is considered in interaction with a quantum particle, the Lorentz force, and the Maxwell equations are obtained. For a quantum particle in a central gravitational field, we obtain the Newtonian acceleration with a correction specific to the Schwarzschild solution, which describes an increase of the gravitational field in the proximity of the gravitational center. We essentially show that a quantum particle is described by a distribution of conservative matter, moving according to the general theory of relativity.

Keywords: Wave packet; Group velocity; Lagrangian; Hamiltonian; Spin; Maxwell equations; Metric tensor; Christoffel symbol; Geodesic.

Introduction

I believe that the physical world will never be described in a fully satisfactory way, for us, as human beings. We shall always feel the necessity to revise this description. In our times more and more authors put into discussion the physical principles [1-10]. We perceive the physical world as a collection of objects in time t, in a three-dimensional space of coordinates $\vec{r} = x\vec{1}_x + y\vec{1}_y + z\vec{1}_z$ where they move with velocities $\vec{v} = \frac{d\vec{r}}{dt}$. For a matter object we define an inertial property called mass, M, and a dynamic quantity as the product of the mass with the velocity, called momentum $\vec{p} = M\vec{v}$. We define also a conservative quantity, called energy, as a sum of the kinetic energy, which depends on momentum, and the potential energy, depending on the coordinates:

$$E = H_0\left(\vec{r}, \vec{p}\right) = T\left(\vec{p}\right) + U\left(\vec{r}\right) \qquad (1)$$

From the conservation condition,

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$$\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{\partial H_0}{\partial \vec{r}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}+\frac{\partial H_0}{\partial \vec{p}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}=0 \qquad (2)$$

we obtain the dynamic equations called Hamilton equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}=\frac{\partial H_0}{\partial \vec{p}}=\frac{\partial}{\partial \vec{p}}T\left(\vec{p}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}=-\frac{\partial H_0}{\partial \vec{r}}=-\frac{\partial}{\partial \vec{r}}U\left(\vec{r}\right) \qquad (3)$$

while the energy function of coordinates and momentum is called Hamiltonian,

$$H_0\left(\vec{r},\vec{p}\right)=\frac{\vec{p}^2}{2M}+U\left(\vec{r}\right) \qquad (4)$$

However, this classical description tells us nothing about the structure of the physical world. Only Quantum Mechanics tells us something about the structure of the physical world. Namely, that this world is composed of species of identical quantum particles. On one hand, experimentally, it has been found that these particles are of a wavy nature. On the other hand, one could find that the simplest way to define a quantum particle is a wave packet, with the momentum conjugated to coordinates, and the energy conjugated to time, and a single quantum constant $\hbar$:

$$\psi_E\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi_0\left(\vec{p},t\right)e^{\frac{\mathrm{i}}{\hbar}\left(\vec{p}\vec{r}-Et\right)}\mathrm{d}^3\vec{p}$$
$$\varphi_0\left(\vec{p},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi_E\left(\vec{r},t\right)e^{-\frac{\mathrm{i}}{\hbar}\left(\vec{p}\vec{r}-Et\right)}\mathrm{d}^3\vec{r} \qquad (5)$$

we can define a momentum operator,

$$\vec{p}=-\mathrm{i}\hbar\frac{\partial}{\partial \vec{r}}\Rightarrow-\mathrm{i}\hbar\frac{\partial}{\partial \vec{r}}\psi_E\left(\vec{r},t\right)=\vec{p}\psi_E\left(\vec{r},t\right) \qquad (6)$$

and a Hamiltonianian operator,

$$H_0=\mathrm{i}\hbar\frac{\partial}{\partial t}=-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial \vec{r}^2}+U\left(\vec{r}\right)=E \qquad (7)$$

acting on these wave functions. A Schrödinger equation is obtained for a particle wave function:

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\vec{r}^2}+U\left(\vec{r}\right)\right]\psi_E\left(\vec{r},t\right)=E\psi_E\left(\vec{r},t\right) \quad (8)$$

however, when the group velocities are calculated for these wave packets,

$$\psi_E\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi_0\left(\vec{p},t\right)e^{\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)+U\left(\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{p}$$
$$\varphi_0\left(\vec{p},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi_E\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)+U\left(\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{r} \quad (9)$$

one obtains an erroneous equation, contradictory to the corresponding Hamilton equation (3) – a minus sign is missing:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}=\frac{\partial H_0}{\partial\vec{p}}=\frac{\partial}{\partial\vec{p}}T\left(\vec{p}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}=\frac{\partial H_0}{\partial\vec{r}}=\frac{\partial}{\partial\vec{r}}U\left(\vec{r}\right) \quad \text{- \underline{erroneous equation}} \quad (10)$$

Relativistic Quantum Principle

We get back the minus sign only if we consider the Lagrangian instead of the Hamiltonian, originally considered in these equations,

$$\psi_0\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi_0\left(\vec{p},t\right)e^{\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{p}$$
$$\varphi_0\left(\vec{p},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi_0\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{r} \quad (11)$$

we notice that these equations, depending on the classical Lagrangian

$$L_0\left(\vec{r},\dot{\vec{r}}\right)=\vec{p}\dot{\vec{r}}-H_0\left(\vec{p},\vec{r}\right)=T\left(\vec{p}\right)-U\left(\vec{r}\right)=\frac{M\vec{v}^2}{2}-U\left(\vec{r}\right) \quad (12)$$

Are still non-realistic, having an infinite spectrum of waves, as a function of the wave velocity. A realistic particle has a finite spectrum as a function of the wave propagation velocity. A finite spectrum is obtained for a relativistic Lagrangian

$$L_0\left(\dot{\vec{r}}\right)\mathrm{d}t = -M_0 c^2 \sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}\mathrm{d}t = -M_0 c\mathrm{d}s \qquad (13)$$

with the momentum

$$\vec{p} = \frac{\partial L_0}{\partial \dot{\vec{r}}} = \frac{M_0 \dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} = M\dot{\vec{r}} \qquad (14)$$

and the mass

$$M = \frac{M_0}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} \qquad (15)$$

As functions of the particle velocity $\vec{v} = \dot{\vec{r}}$ and the cut-off velocity c from (13) we notice that the invariance of the scalar time dependent phase variation of a wave function is equivalent to the invariance of the time-space interval. The invariance of the time-space interval means that a change of coordinates is in fact a rotation of the time-space coordinates.

By a well-known calculation, the relativistic transform of the coordinate intervals is obtained for the quantum particle waves (FIG. 1):

$$\mathrm{d}t = \frac{\mathrm{d}t' + \frac{\mathrm{V}}{\mathrm{c}^2}\mathrm{d}x'}{\sqrt{1-\frac{V^2}{c^2}}}, \quad \mathrm{d}x = \frac{\mathrm{d}x' + \mathrm{V}\mathrm{d}t'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (16)$$

$$\mathrm{d}y = \mathrm{d}y', \quad \mathrm{d}z = \mathrm{d}z'.$$

On this basis, we define a Relativistic quantum principle: The scalar time-dependent phase variation of a quantum particle wave is an invariant for an arbitrary change of coordinates. This means the invariance of the time-space interval, i.e. the relativistic kinematics of the particle waves.

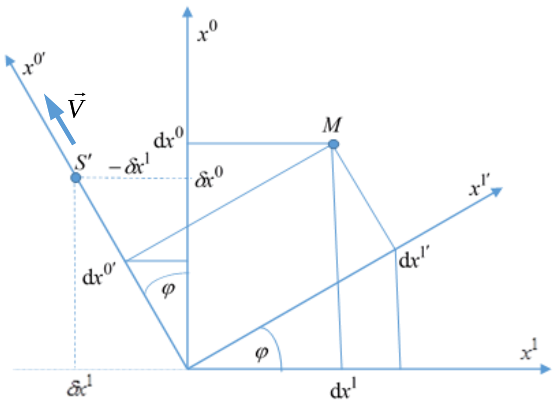

FIG. 1. **Change of coordinates, as a rotation of time-space system $\left(x^{0'}, x^{1'}\right)$, moving with a velocity V in a system $\left(x^0, x^1\right)$.**

Spin-Statistics Relation

For two-particle states $\langle \vec{r}_1, \vec{r}_2 | i_1, i_2 \rangle$, we define the inversion operator I,

$$\langle \vec{r}_2, \vec{r}_1 | i_1, i_2 \rangle = I \langle \vec{r}_1, \vec{r}_2 | i_1, i_2 \rangle \qquad (17)$$

by two successive applications,

$$\langle \vec{r}_1, \vec{r}_2 | i_1, i_2 \rangle = I \langle \vec{r}_2, \vec{r}_1 | i_1, i_2 \rangle = I^2 \langle \vec{r}_1, \vec{r}_2 | i_1, i_2 \rangle \qquad (18)$$

we get the two eigenvalues of this operator corresponding to the Fermi-Dirac statistics, and the Bose-Einstein statistics:

$$I^2 = 1 \begin{cases} I_1 = -1 & \text{- Fermions} \\ I_2 = \ \ 1 & \text{- Bosons} \end{cases} \qquad (19)$$

we take into account that the inversion operation is equivalent to a double rotation of the two particles with an angle π (FIG. 2),

$$I = R_{\pi}^{(1)} R_{\pi}^{(2)} \qquad (20)$$

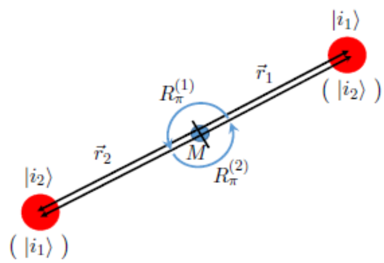

FIG. 2. **A two-particle inversion as a double rotation with an angle π.**

In our case, when no orbital motion is considered, the rotation operator with a differential angle $R_{\delta\vec{\alpha}}$, as a function of the total angular momentum $\vec{J}$, reduces to a function of the proper angular momentum $\vec{S}$,

$$R_{\delta\vec{\alpha}}\psi(\vec{r}) = \psi(\vec{r} + \delta\vec{\alpha}\times\vec{r}) = \psi(\vec{r}) + \delta\vec{\alpha}\times\vec{r}\,\frac{\partial}{\partial\vec{r}}\psi(\vec{r})$$
$$= \psi(\vec{r}) + \delta\vec{\alpha}\cdot\vec{r}\times\frac{\partial}{\partial\vec{r}}\psi(\vec{r}) = e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}\psi(\vec{r}) \qquad (21)$$
$$\mathrm{i}\vec{J} = \mathrm{i}\vec{S}$$

for a rotation angle $\vec{\alpha}$, we obtain the rotation operator

$$R_{\vec{\alpha}} = e^{\mathrm{i}\vec{S}\vec{\alpha}} \qquad (22)$$

which means

$$R_{\pi}^{(1)} = R_{\pi}^{(2)} = e^{\mathrm{i}\pi S} \qquad (23)$$

from (20), we obtain the relation between the eigenvalue of the proper rotation operator, called spin, and the eigenvalue of the inversion operator – the spin-statistics relation:

$$I = e^{\mathrm{i}2\pi S}\begin{cases} I_1 = e^{\mathrm{i}2\pi S_1} = -1 \;\Rightarrow S_1 = \frac{1}{2} & \text{- Fermions} \\ I_2 = e^{\mathrm{i}2\pi S_2} = 1 \;\Rightarrow S_2 = 1 & \text{- Bosons} \end{cases} \qquad (24)$$

Quantum Particle in Electromagnetic Field

When the wave packets

$$\psi\left(\vec{r},t\right)=\frac{1}{\left(2\pi h\right)^{3/2}}\int\varphi\left(\vec{P},t\right)e^{\frac{i}{\hbar}\left[\vec{P}\vec{r}-L\left(\vec{r},\dot{\vec{r}},t\right)t\right]}\mathrm{d}^3\vec{P}$$

$$\varphi\left(\vec{P},t\right)=\frac{1}{\left(2\pi h\right)^{3/2}}\int\psi\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\left[\vec{P}\vec{r}-L\left(\vec{r},\dot{\vec{r}},t\right)t\right]}\mathrm{d}^3\vec{r} \qquad (25)$$

of a quantum particle with a charge e are in a field of a vector potential $\vec{A}\left(\vec{r},t\right)$ and a scalar potential $U\left(\vec{r}\right)$, a time dependent phase variation arises, with terms proportional to variations of coordinates and time:

$$L\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}t=-M_0c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}\mathrm{d}t+e\vec{A}\left(\vec{r},t\right)\mathrm{d}\vec{r}-eU\left(\vec{r}\right)\mathrm{d}t \qquad (26)$$

In this case, we get a canonical momentum $\vec{P}$, which includes a mechanical component $\vec{P}$ as a function of the particle mass and velocity, and an electromagnetic component as the product of the charge with the vector potential:

$$\vec{P}=\frac{\partial}{\partial\dot{\vec{r}}}L\left(\vec{r},\dot{\vec{r}},t\right)=\frac{M_0\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}+e\vec{A}\left(\vec{r},t\right)=\vec{p}+e\vec{A}\left(\vec{r},t\right) \qquad (27)$$

From the group velocity in the momentum space, we get the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\vec{r}}}L\left(\vec{r},\dot{\vec{r}},t\right)=\frac{\partial}{\partial\vec{r}}L\left(\vec{r},\dot{\vec{r}},t\right) \qquad (28)$$

With this equation, for the Hamiltonian

$$H\left(\vec{P},\vec{r},t\right)=\vec{P}\dot{\vec{r}}-L\left(\vec{r},\dot{\vec{r}},t\right) \qquad (29)$$

we obtain a time variation

$$\frac{\mathrm{d}}{\mathrm{d}t}H\left(\vec{P},\vec{r},t\right)=\dot{\vec{P}}\dot{\vec{r}}+\underline{\vec{P}\ddot{\vec{r}}}-\frac{\partial}{\partial\vec{r}}L\left(\vec{r},\dot{\vec{r}},t\right)\dot{\vec{r}}-\underline{\frac{\partial}{\partial\dot{\vec{r}}}L\left(\vec{r},\dot{\vec{r}},t\right)\ddot{\vec{r}}}-\frac{\partial}{\partial t}L\left(\vec{r},\dot{\vec{r}},t\right)$$
$$=\frac{\partial}{\partial\vec{P}}H\left(\vec{P},\vec{r},t\right)\dot{\vec{P}}+\frac{\partial}{\partial\vec{r}}H\left(\vec{P},\vec{r},t\right)\dot{\vec{r}}+\frac{\partial}{\partial t}H\left(\vec{P},\vec{r},t\right),$$

which leads to the Hamilton equations

$$\dot{\vec{r}}=\frac{\partial}{\partial\vec{P}}H\left(\vec{P},\vec{r},t\right)$$
$$\dot{\vec{P}}=-\frac{\partial}{\partial\vec{r}}H\left(\vec{P},\vec{r},t\right)$$
$$\frac{\partial}{\partial t}H\left(\vec{P},\vec{r},t\right)=-\frac{\partial}{\partial t}L\left(\vec{r},\dot{\vec{r}},t\right)\qquad(30)$$

and an explicit time-variation

$$\frac{\mathrm{d}}{\mathrm{d}t}H\left(\vec{P},\vec{r},t\right)=\frac{\partial}{\partial t}H\left(\vec{P},\vec{r},t\right)\qquad(31)$$

It is interesting that the Hamiltonian of a particle in a constant scalar potential $U\left(\vec{r}\right)$, under the action of a time varying vector potential $\vec{A}\left(\vec{r},t\right)$, is a conservative function,

$$H\left(\vec{P},\vec{r},t\right)=\vec{P}\dot{\vec{r}}-L\left(\vec{r},\dot{\vec{r}},t\right)$$
$$=\frac{M_0\dot{\vec{r}}^2}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}+\underline{e\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}}-\left(-M_0c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}+\underline{e\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}}-eU\left(\vec{r}\right)\right)$$

called energy:

$$E\left(\vec{r},\dot{\vec{r}}\right)=H\left(\vec{P},\vec{r},t\right)=\frac{M_0c^2}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}+eU\left(\vec{r}\right)\qquad(32)$$

From (27), we obtain the first term of this expression as a function of the canonical momentum,

$$\frac{M_0^2 c^2}{1-\frac{\dot{\vec{r}}^2}{c^2}} = \frac{M_0^2 \dot{\vec{r}}^2}{1-\frac{\dot{\vec{r}}^2}{c^2}} + M_0^2 c^2 = \left[\vec{P} - e\vec{A}\left(\vec{r}, t\right)\right]^2 + M_0^2 c^2$$

which leads to the canonical form of the Hamiltonian:

$$H\left(\vec{P}, \vec{r}, t\right) = c\sqrt{M_0^2 c^2 + \left[\vec{P} - e\vec{A}\left(\vec{r}, t\right)\right]^2} + eU\left(\vec{r}\right) \qquad (33)$$

From (27), we also obtain the mechanical force acting on a quantum wave,

$$\vec{F}_e = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{P} - e\frac{\mathrm{d}}{\mathrm{d}t}\vec{A}\left(\vec{r}, t\right) \qquad (34)$$

which, with the Lagrange equation (28),

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P} = \frac{\partial}{\partial\vec{r}} L\left(\vec{r}, \dot{\vec{r}}, t\right) = e\frac{\partial}{\partial\vec{r}}\left[\vec{A}\left(\vec{r}, t\right)\dot{\vec{r}}\right] - e\frac{\partial}{\partial\vec{r}} U\left(\vec{r}\right) \qquad (35)$$

and the vector formula

$$\dot{\vec{r}} \times \left[\frac{\partial}{\partial\vec{r}} \times A\left(\vec{r}, t\right)\right] = \frac{\partial}{\partial\vec{r}}\left[\dot{\vec{r}} A\left(\vec{r}, t\right)\right] - \left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right) A\left(\vec{r}, t\right) \qquad (36)$$

takes the Lorentz form,

$$\vec{F}_e = e\vec{E}\left(\vec{r}, t\right) + e\dot{\vec{r}} \times \vec{B}\left(\vec{r}, t\right) \qquad (37)$$

depending on the electric field

$$\vec{E}\left(\vec{r}, t\right) = -\frac{\partial}{\partial\vec{r}} U\left(\vec{r}\right) - \frac{\partial}{\partial t}\vec{A}\left(\vec{r}, t\right) \qquad (38)$$

and the magnetic induction

$$\vec{B}\left(\vec{r}, t\right) = \frac{\partial}{\partial\vec{r}} \times \vec{A}\left(\vec{r}, t\right) \qquad (39)$$

From equations (38) and (39) we obtain the Faraday-Maxwell law of the electromagnetic induction,

$$\frac{\partial}{\partial \vec{r}} \times \vec{E}\left(\vec{r}, t\right) = -\frac{\partial}{\partial t} \vec{B}\left(\vec{r}, t\right) \qquad (40)$$

and the Gauss-Maxwell law of the magnetic induction flow,

$$\frac{\partial}{\partial \vec{r}} \vec{B}\left(\vec{r}, t\right) = 0 \qquad (41)$$

With the gouge condition

$$\frac{\partial}{\partial \vec{r}} \vec{A}\left(\vec{r}, t\right) = 0 \qquad (42)$$

we obtain the Gauss-Maxwell law of the electric field flow,

$$\frac{\partial}{\partial \vec{r}} \vec{E}\left(\vec{r}, t\right) = -\frac{\partial^2}{\partial \vec{r}^2} U\left(\vec{r}\right) = \frac{\rho\left(\vec{r}\right)}{\varepsilon_0} \qquad (43)$$

with the charge density as source of the electric potential $U\left(\vec{r}\right)$, while ε_0 is a physical constant called permittivity. From the time variation of the electric field,

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{E}\left(\vec{r}, t\right) = \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}}\right) \vec{E}\left(\vec{r}, t\right) + \frac{\partial}{\partial t} \vec{E}\left(\vec{r}, t\right) \qquad (44)$$

which, acting on a charge density $\rho\left(\vec{r}\right)$ produces a current density in its direction,

$$\vec{j}\left(\vec{r}, t\right) = \rho\left(\vec{r}, t\right) \dot{\vec{r}} = \sigma \vec{E}\left(\vec{r}, t\right), \; \dot{\vec{r}} \times \vec{E}\left(\vec{r}, t\right) = 0$$

with the vector formula

$$\frac{\partial}{\partial \vec{r}} \times \underbrace{\left[\dot{\vec{r}} \times \vec{E}\left(\vec{r}, t\right)\right]}_{0} = \dot{\vec{r}} \left[\frac{\partial}{\partial \vec{r}} \vec{E}\left(\vec{r}, t\right)\right] - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}}\right) \vec{E}\left(\vec{r}, t\right)$$

$$= \dot{\vec{r}} \frac{\rho\left(\vec{r}, t\right)}{\varepsilon_0} - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}}\right) \vec{E}\left(\vec{r}, t\right) = 0 \qquad (45)$$

we obtain

$$\varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\vec{E}\left(\vec{r},t\right)=\vec{j}\left(\vec{r},t\right)+\varepsilon_0 \frac{\partial}{\partial t}\vec{E}\left(\vec{r},t\right) \qquad (46)$$

Considering for the two fields $\vec{E}$ and $\vec{B}$ an equation symmetric with (40),

$$\varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\vec{E}\left(\vec{r},t\right)=\frac{1}{\mu_0}\frac{\partial}{\partial \vec{r}}\times\vec{B}\left(\vec{r},t\right)$$

which leads to a field propagation with a velocity

$$c=\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad (47)$$

we obtain the Ampère-Maxwell law

$$\frac{1}{\mu_0}\frac{\partial}{\partial \vec{r}}\times\vec{B}\left(\vec{r},t\right)=\vec{j}\left(\vec{r}\right)+\varepsilon_0 \frac{\partial}{\partial t}\vec{E}\left(\vec{r},t\right) \qquad (48)$$

In this way, for the two fields (38) and (39), the Maxwell equations (40), (41), (43) and (48), and the Lorentz force (37), are obtained only from the condition of interaction with a quantum particle. For the physical consistency of this theory, the field propagation velocity (47) is equal to the cutoff velocity c of the quantum particle spectrum (FIG. 3).

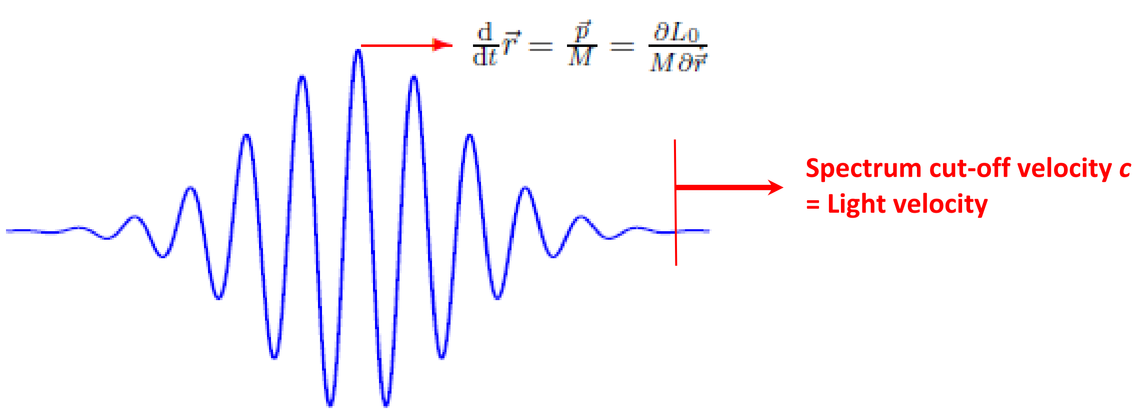

FIG. 3. **Wave packet of a quantum particle with a cutoff velocity of the spectrum equal to the light velocity c.**

The Relativistic Quantum Principle and the Dynamic Equation

We consider the time dependent phase differential

$$\mathrm{d}S = L\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}t = -M_0 c\mathrm{d}s + e\vec{A}\left(\vec{r},t\right)\mathrm{d}\vec{r} - eU\left(\vec{r}\right)\mathrm{d}t \qquad (49)$$

with a term of interaction as the product of the field potential four-vector

$$\left(A_i\right) = \left(A_x, A_y, A_z, \frac{\mathrm{i}}{c}U\right) \qquad (50)$$

with the coordinate four-vector

$$\left(x_i\right) = \left(x, y, z, \mathrm{i}ct\right) \qquad (51)$$

and the time-space interval

$$\mathrm{d}s = \sqrt{-\mathrm{d}x_i^2} \qquad (52)$$

We obtain the time dependent phase of a quantum particle in an electromagnetic field potential as an action

$$S = \int L\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}t = \int\left(-M_0 c\mathrm{d}s + eA_i\mathrm{d}x_i\right) \qquad (53)$$

According to the relativistic quantum principle the time dependent phase variation is null:

$$\delta S = \int\left(M_0 c\frac{\mathrm{d}x_i}{\mathrm{d}s}\delta\mathrm{d}x_i + eA_i\delta\mathrm{d}x_i + e\delta A_i\mathrm{d}x_i\right) = 0 \qquad (54)$$

Integrating by parts, for the velocity four-vector

$$u_i = \frac{\mathrm{d}x_i}{\mathrm{d}s} \qquad (55)$$

and the field four-tensor

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \qquad (56)$$

we obtain the dynamic equations

$$M_0 c \frac{\mathrm{d}u_i}{\mathrm{d}s} = eF_{ik} u_k \qquad (57)$$

Or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M \frac{\mathrm{d}x_i}{\mathrm{d}t} \right) = eF_{ik} \frac{\mathrm{d}x_k}{\mathrm{d}s} \qquad (58)$$

as a function of the relativistic mass

$$M = \gamma M_0 , \quad \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad (59)$$

With the elements of the field four-tensor

$$\left(F_{ik} \right) = \begin{pmatrix} 0 & B_z & -B_y & -\frac{\mathrm{i}}{c} E_x \\ -B_z & 0 & B_x & -\frac{\mathrm{i}}{c} E_y \\ B_y & -B_x & 0 & -\frac{\mathrm{i}}{c} E_z \\ \frac{\mathrm{i}}{c} E_x & \frac{\mathrm{i}}{c} E_y & \frac{\mathrm{i}}{c} E_z & 0 \end{pmatrix} \qquad (60)$$

the dynamic equation (58) takes the form of the Lorentz force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{\vec{r}} \right) = e\vec{E} + e\dot{\vec{r}} \times \vec{B} \qquad (61)$$

The Relativistic Quantum Principle and the Field Transformation

In the null variance of the time-dependent phase, we distinguish a mechanical term and an electromagnetic field term:

$$\delta S = \int \left[\underbrace{-M_0 c \mathrm{d}u_i \delta x_i}_{\text{mechanical}} + \underbrace{F_{ik} u_k \delta x_i \mathrm{d}s}_{\text{electromagnetic}} \right] = 0 \qquad (62)$$

From the invariance of the mechanical term for a change of coordinates,

$$x_i = \alpha_{ij} x'_j \qquad (63)$$

$$\mathrm{d}u_i \delta x_i = \alpha_{ij} \alpha_{ik} \mathrm{d}u'_j \delta x'_k = \alpha_{ji}^{-1} \alpha_{ik} \mathrm{d}u'_j \delta x'_k = \delta_{jk} \mathrm{d}u'_j \delta x'_k = \mathrm{d}u'_j \delta x'_j \qquad (64)$$

we deduce that also the electromagnetic term is an invariant,

$$F_{ik} u_k \delta x_i = F_{ik} \alpha_{kl} u'_l \alpha_{ij} \delta x'_j = F'_{jl} u'_l \delta x'_j \qquad (65)$$

which means the field transformation

$$F'_{jl} = F_{ik} \alpha_{ij} \alpha_{kl} \ \underline{\text{or}} \ F_{ik} = \alpha_{ij} \alpha_{kl} F'_{jl} \qquad (66)$$

For a coordinate transformation (16),

$$\left(\alpha_{ij}\right) = \begin{pmatrix} \gamma & 0 & 0 & -\mathrm{i}\gamma \frac{V}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{i}\gamma \frac{V}{c} & 0 & 0 & 1 \end{pmatrix} \qquad (67)$$

we obtain the field transformation

$$E_x = E_{x'}, \quad E_y = \frac{E_{y'} + VB_{z'}}{\sqrt{1-\frac{V^2}{c^2}}}, \quad E_z = \frac{E_{z'} - VB_{y'}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$B_z = B_{z'}, \quad B_y = \frac{B_{y'} - \frac{V}{c^2}E_{x'}}{\sqrt{1-\frac{V^2}{c^2}}}, \quad B_z = \frac{B_{z'} + \frac{V}{c^2}E_{y'}}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (68)$$

The Spin, As a Characteristic of the Quantum Particle Dynamics

From (25) with the Hamiltonian (29), for a quantum particle in electromagnetic field at a rather low velocity, we obtain a wave packet of the form

$$\psi(\vec{r},t) = \frac{1}{(2\pi h)^{3/2}} \int \varphi(\vec{P},t) e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\vec{r} - \left[\vec{P}\dot{\vec{r}} - H(\vec{P},\vec{r})\right]t\right\}} \mathrm{d}^3\vec{P}$$

$$\approx \frac{1}{(2\pi h)^{3/2}} \int \varphi(\vec{P},t) e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}\vec{r} + Et\right)} \mathrm{d}^3\vec{P} \qquad (69)$$

which means a Schrödinger type equation

$$H(\vec{P},\vec{r})\psi(\vec{r},t) = E\psi(\vec{r},t) \qquad (70)$$

with the Hamiltonian (33), with the Dirac operators

$$\alpha_0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \qquad (71)$$

is of the form

$$H(\vec{p},\vec{r}) = c\sqrt{M_0^2c^2 + \vec{p}^2} + eU(\vec{r}) = c\left(\alpha_0 M_0 c + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3\right) + eU(\vec{r}) \qquad (72)$$

We obtain the Schrödinger-Dirac equation

$$\left[c\left(\alpha_0 M_0 c + \vec{\alpha}\vec{p}\right) + eU(\vec{r})\right]\psi(\vec{r},t) = E\psi(\vec{r},t) \qquad (73)$$

with the notation

$$\vec{\alpha} = \left(\alpha_1, \alpha_2, \alpha_3\right) \qquad (74)$$

The Dirac operators, with the anti-commutation relations

$$\left\{\alpha_i, \alpha_j\right\} = \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \qquad (75)$$

are functions of the Pauli spin operators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (76)$$

with the anti-commutation relations

$$\left\{\sigma_i, \sigma_j\right\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \qquad (77)$$

while the particle wave function is split into components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \qquad (78)$$

From (73) with (71) and (78), we obtain a two-dimensional Schrödinger equation

$$c\left[\begin{pmatrix} \psi_1(\vec{r}) \\ -\psi_2(\vec{r}) \end{pmatrix} M_0 c + \begin{pmatrix} \vec{\sigma}\psi_2(\vec{r}) \\ \vec{\sigma}\psi_1(\vec{r}) \end{pmatrix} \vec{p}\right] + eU(\vec{r}) \begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix} = E \begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix} \qquad (79)$$

with the notation

$$\vec{\sigma} = \left(\sigma_1, \sigma_2, \sigma_3\right) \qquad (80)$$

which for the two components is

$$\left[M_0c^2+eU\left(\vec{r}\right)\right]\ \psi_1\left(\vec{r}\right)+c\vec{\sigma}\vec{p}\psi_2\left(\vec{r}\right)=E\psi_1\left(\vec{r}\right)$$
$$\left[-M_0c^2+eU\left(\vec{r}\right)\right]\psi_2\left(\vec{r}\right)+c\vec{\sigma}\vec{p}\psi_1\left(\vec{r}\right)=E\psi_2\left(\vec{r}\right) \qquad (81)$$

In a non-relativistic approximation, small velocity, small potential, which means $E\approx M_0c^2$

$$\underbrace{\left[E+M_0c^2-eU\left(\vec{r}\right)\right]}_{2M_0c^2}\left[\underbrace{E-M_0c^2}_{E_c}-eU\left(\vec{r}\right)\right]\ \psi_1\left(\vec{r}\right)=c^2\left(\vec{\sigma}\vec{p}\right)^2\psi_1\left(\vec{r}\right)$$
$$\left[\underbrace{E-M_0c^2}_{E_c}-eU\left(\vec{r}\right)\right]\underbrace{\left[E+M_0c^2-eU\left(\vec{r}\right)\right]}_{2M_0c^2}\psi_2\left(\vec{r}\right)=c^2\left(\vec{\sigma}\vec{p}\right)^2\psi_2\left(\vec{r}\right)$$

and the classical energy

$$E_c=E-M_0c^2 \qquad (82)$$

the two wave function components satisfy the Schrödinger-Dirac equations:

$$\left[\frac{\left(\vec{\sigma}\vec{p}\right)^2}{2M_0}+eU\left(\vec{r}\right)\right]\psi_1\left(\vec{r}\right)=E_c\psi_1\left(\vec{r}\right)$$
$$\left[\frac{\left(\vec{\sigma}\vec{p}\right)^2}{2M_0}+eU\left(\vec{r}\right)\right]\psi_2\left(\vec{r}\right)=E_c\psi_2\left(\vec{r}\right) \qquad (83)$$

With the mechanical momentum from equation (27),

$$\vec{p}=\vec{P}-e\vec{A}\left(\vec{r},t\right)=-i\hbar\nabla-e\vec{A}\left(\vec{r},t\right) \qquad (84)$$

for the last term of

$$\left(\vec{\sigma}\vec{p}\right)^2 = \left(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3\right)^2 = \vec{p}^2 + \mathrm{i}\vec{\sigma}\left(\vec{p}\times\vec{p}\right) \qquad (85)$$

we find

$$\left(\vec{p}\times\vec{p}\right)\psi = \mathrm{i}e\hbar\left(\nabla\times\vec{A}+\vec{A}\times\nabla\right)\psi = \mathrm{i}e\hbar\vec{B}\psi \qquad (86)$$

With these expressions,

$$\left(\vec{\sigma}\vec{p}\right)^2 = \vec{p}^2 - e\hbar\vec{\sigma}\vec{B}$$

and the Schrödinger-Dirac equations (83) take a form

$$\left[\frac{\vec{p}^2}{2M_0} - \vec{\mu}_S\vec{B} + eU\left(\vec{r}\right)\right]\psi_1\left(\vec{r}\right) = E_C\psi_1\left(\vec{r}\right)$$
$$\left[\frac{\vec{p}^2}{2M_0} - \vec{\mu}_S\vec{B} + eU\left(\vec{r}\right)\right]\psi_2\left(\vec{r}\right) = E_C\psi_2\left(\vec{r}\right) \qquad (87)$$

depending on the spin magnetic momentum

$$\vec{\mu}_S = \frac{\hbar e}{2M_0}\vec{\sigma} \qquad (88)$$

with the component

$$\mu_3 = \frac{\hbar e}{2M_0} \qquad (89)$$

in the direction of the magnetic field. This moment appears as an effect of a proper rotational motion with an angular momentum s_3. We take into account that the total angular momentum j_3, including also the orbital momentum l_3, commutes with the Hamiltonian,

$$\left[H, j_3\right] = \left[H, l_3 + s_3\right] = 0 \qquad (90)$$

which means that

$$\left[H, s_3\right] = -\left[H, l_3\right] \qquad (91)$$

With the commutation relations

$$\left[p_i, l_j\right] = \mathrm{i}\hbar\delta_{\mathrm{ijk}} p_k \qquad (92)$$

from (91) with the Hamiltonian (72), we obtain the equations

$$\begin{aligned} \left[\alpha_1, s_3\right] &= -\mathrm{i}\hbar\alpha_2 \\ \left[\alpha_2, s_3\right] &= \mathrm{i}\hbar\alpha_1 \\ \left[\alpha_3, s_3\right] &= 0 \\ \left[\alpha_4, s_3\right] &= 0 \end{aligned} \qquad (93)$$

With the anti-commutation relations (75), we find the solution

$$s_3 = s\alpha_1\alpha_2 \qquad (94)$$

Where

$$s = -\mathrm{i}\frac{\hbar}{2} \qquad (95)$$

With the expressions (71) of the Dirac operators, we find the proper angular momentum

$$s_3 = -\mathrm{i}\frac{\hbar}{2}\alpha_1\alpha_2 = \frac{\hbar}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \qquad (96)$$

with the eigenvalues

$$s_3 = \pm\frac{\hbar}{2} \qquad (97)$$

From (89) and (97) we find the gyromagnetic ratio

$$g_s = \frac{\mu_3}{s_3} = \frac{e}{M_0} \qquad (98)$$

Dynamics of a Quantum Particle in a Gravitational Field

In the framework of the general theory of relativity [11], we consider a generalization of the quantum particle wave functions (25) for a curvilinear system of time-space coordinates $x^0 = ct, x^1, x^2, x^3$,

$$\psi\left(x^{\mathrm{i}},t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \varphi\left(\dot{x}^{\mathrm{i}},t\right) e^{\frac{\mathrm{i}}{\hbar}M_0 c\int\left(-g_{ij}\dot{x}^i\dot{x}^{\mathrm{j}}+1\right)\mathrm{d}s} M_0^3\mathrm{c}^3 \frac{\partial\left(\dot{x},\dot{y},\dot{z}\right)}{\partial\left(\dot{x}^1,\dot{x}^2,\dot{x}^3\right)} \mathrm{d}\dot{x}^1\mathrm{d}\dot{x}^2\mathrm{d}\dot{x}^3$$

$$\varphi\left(\dot{x}^{\mathrm{i}},t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \psi\left(x^{\mathrm{i}},t\right) e^{-\frac{\mathrm{i}}{\hbar}M_0 c\int\left(-g_{ij}\dot{x}^i\dot{x}^{\mathrm{j}}+1\right)\mathrm{d}s} \frac{\partial\left(x,y,z\right)}{\partial\left(x^1,x^2,x^3\right)} \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \quad i, j... = 1,2,3 \qquad (99)$$

depending on the time-space interval

$$\mathrm{d}s = \sqrt{g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta}, \quad \alpha,\beta... = 0,1,2,3 \qquad (100)$$

and the time-space velocities

$$\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \qquad (101)$$

which satisfy the relation

$$g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1 \qquad (102)$$

In a classical approximation, small particle, small velocity $\dot{x}^i$, we can consider a linearization of the wave particle phase,

$$\int\left(-g_{ij}\dot{x}^i\dot{x}^j\mathrm{d}s+\mathrm{d}s\right)=\int\left(-g_{ij}\dot{x}^i\mathrm{d}x^j+\sqrt{c^2\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j}\right)$$
$$=\int\left(-g_{ij}\dot{x}^i\mathrm{d}x^j+c\mathrm{d}t\sqrt{1+g_{ij}\dot{x}^i\dot{x}^j}\right)$$
$$=\int\left[-g_{ij}\dot{x}^i\mathrm{d}x^j+c\mathrm{d}t\left(1+\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j\right)\right]$$
$$=\int\left(-\frac{1}{2}g_{ij}\dot{x}^i\mathrm{d}x^j+c\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\mathrm{d}t\right)$$
$$=-\frac{1}{2}g_{ij}\dot{x}^i x^j+c\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\,t$$

In this case, a quantum particle is described by two wave packets of the form

$$\psi\left(x^{\mathrm{i}},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(\dot{x}^{\mathrm{i}},t\right)e^{\frac{\mathrm{i}}{\hbar}M_0c\left(-\frac{1}{2}g_{ij}\dot{x}^i x^{\mathrm{j}}+c\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}t\right)}M_0^3\mathrm{c}^3\frac{\partial\left(\dot{x},\dot{y},\dot{z}\right)}{\partial\left(\dot{x}^1,\dot{x}^2,\dot{x}^3\right)}\mathrm{d}\dot{x}^1\mathrm{d}\dot{x}^2\mathrm{d}\dot{x}^3$$
$$\varphi\left(\dot{x}^{\mathrm{i}},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^{\mathrm{i}},t\right)e^{-\frac{\mathrm{i}}{\hbar}M_0c\left(-\frac{1}{2}g_{ij}\dot{x}^i x^{\mathrm{j}}+c\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}t\right)}\frac{\partial\left(x,y,z\right)}{\partial\left(x^1,x^2,x^3\right)}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \qquad (103)$$

with a group velocity

$$\left\langle v^j\right\rangle=\left\langle\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\rangle=2\left\langle\frac{\partial}{\partial\left(g_{ij}\dot{x}^i\right)}\left(c\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\right)\right\rangle=2c\left\langle\frac{\dot{x}_j}{2\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right\rangle=c\left\langle\frac{\mathrm{d}x^j}{\mathrm{d}s}\right\rangle \qquad (104)$$

In a gravitational field V , the time-time metric element takes the form

$$g_{00}=1-2V \qquad (105)$$

while the particle gets an acceleration on a geodesic,

$$\left\langle a^j\right\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\left\langle v^j\right\rangle=c\left\langle\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2}\frac{\mathrm{d}s}{\mathrm{d}t}\right\rangle=c^2\left\langle\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2}\right\rangle=-c^2\Gamma^j_{\mu\nu}\left\langle\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right\rangle \qquad (106)$$

Introducing the first kind Christoffel symbol, and taking into account the expression of this symbol as a function of the metric tensor elements,

$$\left\langle a^j \right\rangle = -c^2 \Gamma^j_{\mu\nu} \left\langle \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \right\rangle = -c^2 g^{j\lambda} \Gamma_{\lambda\mu\nu} \left\langle \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \right\rangle$$

$$= -c^2 g^{j\lambda} \frac{1}{2} \left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \right) \left\langle \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \right\rangle, \qquad (107)$$

with the Schwarzschild solution for the time-space interval in spherical coordinates, $x^1 = r, x^2 = \theta, x^3 = \varphi$,

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\varphi^2$$

we obtain

$$\left\langle a^j \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \left\langle v^j \right\rangle = -c^2 g^{j\lambda} \frac{1}{2} \left(\underset{0}{g_{\lambda 0,0}} + \underset{0}{g_{\lambda 0,0}} - g_{00,\lambda} \right) \underset{1}{\frac{\mathrm{d}x^0}{\mathrm{d}s}} \underset{1}{\frac{\mathrm{d}x^0}{\mathrm{d}s}}$$

$$= c^2 g^{j\lambda} \frac{1}{2} g_{00,\lambda} = c^2 g^{j\lambda} \frac{1}{2} \frac{\partial}{\partial x^\lambda} \left(1 - 2V\right)$$

$$= -c^2 g^{j\lambda} \frac{\partial V}{\partial x^\lambda} = -c^2 g^{j\lambda} \frac{\partial}{\partial x^\lambda} \left(\frac{m}{r}\right)$$

$$= -mc^2 g^{j1} \frac{\partial}{\partial r} \left(\frac{1}{r}\right)$$

Taking into account that in a central field our particle is accelerated only in the radial direction, we obtain,

$$\left\langle a^1 \right\rangle = -\left(1 - \frac{2m}{r}\right)^{-1} \frac{mc^2}{r^2} \quad \begin{cases} \text{- Newtonian field with rwlativistic correction} \\ \text{(Scwarzschild metrics)} \end{cases}$$

$$\left\langle a^2 \right\rangle = 0$$

$$\left\langle a^3 \right\rangle = 0 \qquad (108)$$

Quantum Particle Wave as a Distribution of Matter

We consider a quantum particle described by wave functions of the form (99) or (103) as a distribution of matter with a normalized density $\rho(x, y, z, t)$ in Cartesian coordinates,

$$\int \rho\left(x, y, z, t\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\left|\psi\left(x, y, z, t\right)\right|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = 1 \qquad (109)$$

or $\rho\left(x^i, t\right)$ in curvilinear coordinates,

$$\int \rho\left(x^i, t\right)\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \int\left|\psi\left(x^i, t\right)\right|^2 \frac{\partial\left(x, y, z\right)}{\partial\left(x^1, x^2, x^3\right)}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = 1 \qquad (110)$$

satisfying the relation

$$\rho\left(x^i, t\right) = \left|\psi\left(x^i, t\right)\right|^2 \frac{\partial\left(x, y, z\right)}{\partial\left(x^1, x^2, x^3\right)} \qquad (111)$$

We define a velocity field of the matter motion

$$v^j = \frac{\mathrm{d}x^j}{\mathrm{d}t} = c\frac{\mathrm{d}x^j}{\mathrm{d}s} = c\dot{x}^j \qquad (112)$$

From the invariance of the time-space interval,

$$\mathrm{d}s = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}$$

which is

$$1 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

by covariant derivation,

$$0 = \left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)_{:\sigma} = g_{\mu\nu}\left(\dot{x}^\mu\dot{x}^\nu_{:\sigma} + \dot{x}^\mu_{:\sigma}\dot{x}^\nu\right) = 2g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu_{:\sigma}$$

we find that any covariant derivation of the velocity is perpendicular to this velocity, as a condition of the matter motion:

$$\dot{x}_\nu\dot{x}^\nu_{:\sigma} = 0 \qquad (113)$$

This means that, besides the geodesic acceleration, we suppose an additional acceleration component A^μ,

$$\frac{\mathrm{d}\dot{x}^{\mu}}{\mathrm{d}s} = \dot{x}^{\mu}_{,\nu}\dot{x}^{\nu} = -\Gamma^{\mu}_{\nu\sigma}\dot{x}^{\nu}\dot{x}^{\sigma} + A^{\mu}$$

which is

$$A^{\mu} = \left(\dot{x}^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\sigma}\dot{x}^{\sigma}\right)\dot{x}^{\nu}$$

By the scalar multiplication with the velocity $\dot{x}_{\mu}$,

$$A^{\mu} = \dot{x}^{\mu}_{;\nu}\dot{x}^{\nu} / \dot{x}_{\mu}$$

from (113) we find that any possible additional acceleration A^{μ} of a particle wave is perpendicular to the velocity $\dot{x}_{\mu}$ of this wave:

$$\dot{x}_{\mu}A^{\mu} = \underbrace{\dot{x}_{\mu}\dot{x}^{\mu}_{;\nu}}_{0}\,\dot{x}^{\nu} = 0 \qquad (114)$$

In other words, the theory of general relativity describes a motion of the matter in planes, perpendicular to the geodesic motion, according to the wavy description (99) or (103) of this motion, as it is illustrated in FIG. 4 .

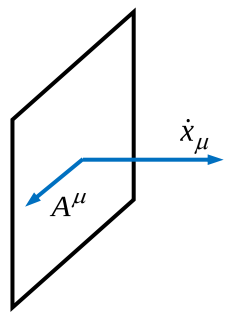

FIG. 4. **Wave plane in the propagation of a quantum particle as a continuous matter motion, according to the general theory of relativity.**

At the same time, we notice that the transformation of the normalization condition of the matter density for an arbitrary system of coordinates,

$$\int \rho\left(x^{\mu'}\right) \mathrm{d}x^{0'} \mathrm{d}x^{1'} \mathrm{d}x^{2'} \mathrm{d}x^{3'} = \int \rho\left(x^{\mu}\right) \underline{J} \mathrm{d}x^{0} \mathrm{d}x^{1} \mathrm{d}x^{2} \mathrm{d}x^{3} \qquad (115)$$

includes the Jacobian

$$J = \mathrm{Det}\left(x^{\mu'}_{,\alpha}\right) = \frac{\partial\left(x^{0'}, x^{1'}, x^{2'}, x^{3'}\right)}{\partial\left(x^{0}, x^{1}, x^{2}, x^{3}\right)} \qquad (116)$$

as a determinant of the tensor transformation elements. From the tensor transformation of the metric tensor,

$$g_{\alpha\beta} = x^{\mu'}_{,\alpha} x^{\nu'}_{,\beta} g_{\mu'\nu'} \qquad (117)$$

the Jacobian is obtained as a ratio of the square roots of the metric tensor determinants

$$g = \mathrm{Det}\left(g_{\alpha\beta}\right) \qquad (118)$$

for the two systems of coordinates:

$$J = \frac{\sqrt{-g}}{\sqrt{-g'}} \qquad (119)$$

With this expression, from the density normalization condition (115), we obtain a density invariant for an arbitrary change of coordinates:

$$\rho\left(x^{\alpha}\right)\sqrt{-g} = \rho\left(x^{\mu'}\right)\sqrt{-g'} = \text{Invariant} \qquad (120)$$

For the matter flow vector

$$J^{\mu} = \rho \dot{x}^{\mu} \qquad (121)$$

we consider a null covariant divergence:

$$J^{\mu}_{\ :\mu} = J^{\mu}_{,\mu} + \Gamma^{\mu}_{\nu\mu} J^{\nu} = J^{\nu}_{,\nu} + \Gamma^{\mu}_{\nu\mu} J^{\nu} = 0 \qquad (122)$$

From the expression

$$\Gamma^{\mu}_{\nu\sigma} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma} = g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\sigma\nu,\lambda}\right) \qquad (123)$$

we obtain the Christoffel symbol in equation (122) as a function of the metric tensor determinant,

$$\Gamma^{\mu}_{\nu\mu} = \frac{1}{2}\underline{\underline{g^{\mu\lambda}}}\left(\underline{g_{\lambda\nu,\mu}} + g_{\lambda\mu,\nu} - \underline{g_{\mu\nu,\lambda}}\right) = \frac{1}{2}g^{\mu\lambda}g_{\lambda\mu,\nu} = \frac{1}{2}g^{-1}g_{,\nu}$$
$$= \frac{1}{2}\left(-g\right)^{-1}\left(-g\right)_{,\nu}$$

With the expression

$$\frac{\left(-g\right)_{,\nu}}{2\left(-g\right)} = \frac{\left(\sqrt{-g}\right)_{,\nu}}{\sqrt{-g}}$$

this Christoffel symbol is

$$\Gamma^{\mu}_{\nu\mu} = \frac{\left(\sqrt{-g}\right)_{,\nu}}{\sqrt{-g}} \qquad (124)$$

With this expression, equation (122) takes the form of a null divergence

$$J^{\mu}{}_{:\mu}\sqrt{-g} = J^{\nu}_{,\nu}\sqrt{-g} + J^{\nu}\underbrace{\Gamma^{\mu}_{\nu\mu}\sqrt{-g}}_{\left(\sqrt{-g}\right)_{,\nu}} = \left(J^{\nu}\sqrt{-g}\right)_{,\nu} = 0$$

which is

$$J^{\mu}{}_{:\mu}\sqrt{-g} = \left(J^{\mu}\sqrt{-g}\right)_{,\mu} = 0 \qquad (125)$$

In this way, we obtain a null integral of the matter flow divergence over a volume V of the space coordinates:

$$\int_V \left(J^{\mu}\sqrt{-g}\right)_{,\mu} \mathrm{d}^3x = 0 \qquad (126)$$

In this integral, we can separate the time derivative term from the space derivative terms, which means a matter conservation relation,

$$\left(\int_V J^0 \sqrt{-g}\mathrm{d}^3 x\right)_{,0} = -\int_V \left(J^m \sqrt{-g}\right)_{,m} \mathrm{d}^3 x$$
$$= -\oint_{\Sigma_V} J^m \sqrt{-g}\mathrm{d}^2 x, \quad m = 1,2,3 \qquad (127)$$

as a time variation of the matter contained in a volume V by a matter flow through the surface Σ_V of this volume. We notice that for the nonrelativistic case, $\dot{x}^m << c$,

$$J^0 = \rho \dot{x}^0 \approx \rho$$
$$J^m = \rho \dot{x}^m \qquad (128)$$

while the metric tensor determinant is $g = -1$. In this way, we obtain the classical expression of the conservation condition:

$$\int_V \rho \mathrm{d}^3 \vec{r} = -\oint_{\Sigma_V} \vec{J} \mathrm{d}^2 \vec{r} \qquad (129)$$

In this way, a quantum particle appears as a distribution of continuous conservative matter, moving on geodesics, with possible accelerations only perpendicularly to these geodesics, in agreement with the wavy description of the quantum mechanics.

Conclusion

We formulated a relativistic quantum principle, asserting that a quantum particle is described by a wave packet with a finite spectrum, and invariant time dependent phases to an arbitrary change of coordinates. Based on this principle, we derived a unitary relativistic quantum theory, including:

- The relativistic kinematics of a quantum particle
- The relativistic dynamics of a quantum particle
- The Maxwell equations of a field interacting with a quantum particle
- The dynamics of a quantum particle in electromagnetic field
- The relativistic electrodynamics
- The spin of a quantum particle
- The relativistic dynamics of a particle in a gravitational field, as a deformation of the Time-space metrics.

We essentially showed that a quantum particle can be described as a distribution of continuous conservative matter moving according to the general theory of relativity.

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