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ANALYTICITY THEOREM FOR FRACTIONAL LAPLACE TRANSFORM

P. R. DESHMUKH* and A. S. GUDADHE[a]

Prof. Ram Meghe Institute of Technology & Research, Badnera, AMRAVATI (M.S.) INDIA
[a]Department of Mathematics, Government Vidarbha Institute of Science & Humanities, AMRAVATI (M.S.) INDIA

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ABSTRACT

Canonical transformation when applied to a quantum mechanics, in a group need to be extended from real domain to complex. Torre A. had introduced fractional Laplace transform a special case of linear canonical transform with complex entries in the representative matrix. The fractional Laplace transform (FrLT) has been used in several areas including signal processing, optical communication etc. In this paper we have proved the analyticity theorem for the generalized fractional Laplace transform.

Key words: Linear canonical transform**,** Fractional Laplace transform

INTRODUCTION

Now a day, the fractional integral transform plays an important role in signal processing, image construction, optical communication, filter design etc.[5,6] Fourier transform is one of the most frequently used tool in signal processing and in many other disciplines. Namies[2] introduced the fractional Fourier transform. Bhosale[1] had extended fractional Fourier transform to the distribution of compact support. Torre[4] had also introduced fractional Laplace transform as a special case of complex linear canonical transform when characteristic matrix is $\begin{bmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{bmatrix}$. The operational transform formulae have been derived in[3].

In this paper we have defined generalized fractional Laplace transform. The Analyticity theorem on fractional Laplace transform is proved in section 2. Lastly conclusion is given in section 3.

Fractional Laplace transform

The fractional Laplace transform is a generalization of Laplace transform. It is a powerful tool for the analysis of time varying functions. The properties and applications of the conventional Laplace transform are special case of fractional Laplace transform.

The fractional Laplace transform of the square integrable function $f(t)$ is defined in terms of a kernel, as –

Available online at www.sadgurupublications.com
*Author for correspondence; E-mail: pdeshmukh92@gmail.com, alka.gudadhe@gmail.com

$$L^{\propto}(u) = \int_{-\infty}^{\infty} f(t)\, K_{\phi}\,(t,u)\, dt$$

where the kernel $K_\phi\,(t,u)$ of the fractional Laplace transform is given by –

$$K_\phi\,(t,u) = \sqrt{\frac{1-icot\phi}{2\pi e}}\; e^{\frac{\tau^2}{2}cot\phi + \frac{u^2}{2}cot\phi - \tau u csc\phi}, \phi \text{ is not multiple of } \pi$$

$$= \delta\,(\text{t-u}),\ \phi \text{ is a multiple of } \pi$$

Moreover, the fractional Laplace transform with $\propto = 1$, corresponds to classical Laplace transform and with $\propto = 0$, corresponds to the identity operator.

Analyticity on fractional Laplace transform

The Generalized Fractional Laplace Transform on E'

Let $Sa = \{t : t\in R^n, | t | \leq a, a > 0\}$ and $t\in$ Sa, $0 \leq a < \infty$, if

$$K_\phi\,(t,\ u) = \sqrt{\frac{1-icot\phi}{2\pi i}}\; e^{\frac{\tau^2}{2}cot\phi + \frac{u^2}{2}cot\phi - \tau u csc\phi},\ \text{where } \phi = \propto\frac{\pi}{2}$$

then $K_\phi\,(t,\ u)\in E(R^n)$, if $\gamma_k\,\{K_\phi\,(t,\ u)\} = \sup_{-\infty<\tau<\infty} | D^k K_\phi\,(t,\ u) | < \infty$

Hence $E\,(R^n)$ is the testing function space. The generalized fractional Laplace transform of $f\,(t)\in E'\,(R^n)$ is defined as –

$L^{\propto}\,\{f\,(t)\}\,(u) = \{f\,(t), K_\phi\,(t,\ u)\}$, where $E'\,(R^n)$ is the dual space of the testing function space.

Analyticity theorem

Let $f\in E'\,(R^n)$ and $L^{\propto}(u) = \{f\,(t)\}, K_\phi\,(t,\ u)\}$, then $L^{\propto}[f\,(t)]\,(u)$ is analytic on C^n if the supp $f\subset Sa = \{t : t\in R^n, | t | \leq a, a > 0\}$ and for each $\varepsilon > 0$, there exist a constant 'C_1' and a positive integer K, such that $0 < \phi \leq \frac{\pi}{2}$, $| L^{\propto}(u)| \leq C_1 C_{1\phi}\, C_{2\phi}^{k}$

$$| L^{\propto}(u)| \leq C_1 C_{1\phi}\, C_{2\phi}^{k}\,(1 + 2(acos\phi + |u|\})^K\, e\ \{C_2\phi((a + \varepsilon)^2 + |I_m u|^2)\, cos\phi\text{-}2(a + \varepsilon)|I_m u|\}$$

Moreover $L^{\propto}(u)$ is differentiable and $D_u^k L^{\propto}(u) = \{f\,(t), D_u^k K_\phi\,(t,u)\}$.

Proof: Let $u = (u_1, u_2, u_3, \ldots u_i, \ldots u_n) \in C^n$

First we prove it for $k = 1$, that is we show,

$$\frac{\partial}{\partial u_i} L^{\propto}(u) = \left\{ f\,(t), \frac{\partial}{\partial u_i} K_\phi\,(t,u) \right\}$$

The general result follows by induction

For fixed $u_i \neq 0$ choose two concentric circles C and C' with center u_i and radius r and r_1 respectively, such that $0 < r < r_1 < |u_i\}$.

Let Δ_i be a complex increment such that $0 < |\Delta_i| < r$. Now consider

$$\frac{L^{\propto}(u_i+\Delta u_i)-L^{\propto}(u_i)}{\Delta u_i}-\left\{f(t),\frac{\partial}{\partial u_i}K\phi(t,u)\right\}=\{f(t),\boldsymbol{\psi}_{\Delta u_i}(t)\} \quad \ldots(2.2.1)$$

where $\boldsymbol{\psi}_{\Delta u_i}(t)=\frac{1}{\Delta u_i}\{K_\phi(t,u_1,u_2,\ldots u_i+\Delta u_i,\ldots u_n)-K_\phi(t,u))-\frac{\partial}{\partial u_i}K_\phi(t,u)$

and $L^{\propto}(u) = \int_{-\infty}^{\infty} f(t)K_\phi(t,u)dt,$

$$= \sqrt{\frac{1-icot\phi}{2\pi i}}\int_{-\infty}^{\infty} e^{\frac{\tau^2}{2}cot\phi+\frac{u^2}{2}cot\phi-\tau ucsc\phi} f(t)\,dt$$

Where $K_\phi(t, u) = \sqrt{\frac{1-icot\phi}{2\pi i}}\; e^{\frac{\tau^2}{2}cot\phi+\frac{u^2}{2}cot\phi-\tau ucosec\phi}$

$$= \sqrt{\frac{\sin\phi}{2\pi\, isin\phi}}\; e^{\frac{1}{2\sin\phi}[(\tau)]^2 cot\phi+u^2\, cot\phi-2\tau u)}$$

$$= \sqrt{\frac{-ie^{i\phi}}{2\pi\, isin\phi}}\; e^{\frac{1}{2\sin\phi}[(\tau)]^2 cot\phi+u^2\, cos\phi-2\tau u)}$$

$$= \left[(2\pi i\sin\phi)^{-\frac{1}{2}}ie^{i\frac{\phi}{2}}\right]e^{\frac{1}{2\sin\phi}[(\tau)]^2 cos\phi+u^2\, cos\phi-2\tau u)}$$

$$= C_1\phi\, e^{C_{2\phi}[(\tau)]^2 cos\phi+u^2\, cos\phi-2\tau u)} \quad \ldots(2.2.2)$$

For any $t\in R^n$ and every fixed integer $K = (k_1, k_2, \ldots .k_n,) \in N_0^{\,n}$

$$\therefore\; D_\tau^k K_\phi(t,u) = D_\tau^k\{C_{1\phi}e^{C_{2\phi}[(\tau)]^2\cos\phi+u^2\cos\phi-2\tau u)}\}$$

$$= C_{1\phi}\sum_{vsK}(2C_{2\phi})^{K-v}\binom{K}{v}(\cos\phi)^v C_v\,(t\cos\phi-u)^{K-2v}K_\phi(t,u)$$

$$= \sum_{v=0}^{Ks2\mu}CvC\phi\sum_{vsK}(t\cos\phi-u)^{K-2v}K_\phi(t,u) \quad \ldots(2.2.3)$$

Where $C_\phi = C_{1\phi}(2C_{2\phi})^{K-v}\binom{K}{v}(\cos\phi)^v$

C_v and C_ϕ are the constant depending upon ϕ, k, v.

Since for any $t\in R^n$, fixed integer k and $\propto$ from, o to $\frac{\pi}{2}$, $D_\tau^k K_\phi(t,u)$ is analytic inside and on C',

$\therefore$ By using Cauchy integral formula –

$$D_\tau^k \psi \Delta_{u_i}(t) = \frac{1}{2\pi i} D_\tau^k \oint_{C'} K_\phi(t,u) \left\{ \frac{1}{\Delta_{u_i}} \left(\frac{1}{z-u_i-\Delta u_i} - \frac{1}{z-u_i} \right) - \frac{1}{(z-u_i)^2} \right\} dz$$

$$= \frac{\Delta_{u_i}}{2\pi i} \oint_{C'} \frac{D_\tau^k K_\phi(t,u)}{(z-u_i-\Delta u_i)(z-u_i)^2} dz$$

$$= \frac{\Delta_{u_i}}{2\pi i} \oint_{C'} \frac{M\ (t,u)}{(z-u_i-\Delta u_i)(z-u_i)^2} dz$$

$$|z-u_i-\Delta u_i| > (r_1 - r) > 0 \quad \ldots(2.2.3)$$

and $|z - u_i| = r_1$

But for all $z \in C_1$ and t restricted to a compact subset of R^n, $o \le \propto \le \frac{\pi}{2}$

$M\,(t, u)$ is bounded by a constant.

$$|D_\tau^k \psi_{\Delta u_i}(t)| \le |\Delta u_i| \frac{K}{(r_1 - r) r_1}$$

As $|\Delta u_i| \to 0, |D_\tau^k \psi_{\Delta u_i}(t)| \to 0$ uniformly on compact subset of R^n.

$\psi_{\Delta u_i}(t)$ Converges in $E\,(R^n)$ to zero. Since $f \in E'$, we conclude that (2.2.1) tends to zero.

$\therefore$ $L^\propto$ (u) is differentiable with respect to u_i and it is true for all I = 1,2,3…..,n.

Hence $L^\propto$ (u) is analytic on C^n and $D_u^k L^\propto(u) = \{f(t),\ D_u^k K\,(t,u)\}$.

To prove the second part

supp $f \subset S_a$ and let $\varepsilon > 0$ choose $\rho \in D\,(R^n)$, such that $\rho(t) = 1$ on the neighborhood of S_a and supp $\rho \subset S_{a+E'}$ then by boundedness property of the generalized function there exist a constant C and a non-negative integer K such that –

$$|\,[(L)]^{\uparrow} \propto (u)\,| \le |\,\langle f(t), - K_{\downarrow}\, \phi\,(t, u) \rangle\,|$$

$$\le C \max_{|\beta| \le K} \sup_{\tau \in R^n} \left| D_\tau^\beta [\rho(t) K_\phi\,(t,u)] \right|$$

$$\le C \max_{|\beta| \le K} \sup_{\tau \in R^n} \sum_\nu \nu^\beta \left| D_\tau^{\beta-\nu} [\rho(t) D^\nu K_\phi\,(t,u)] \right|$$

$$\le C \max_{|\beta| \le K} \sup_{\tau \in R^n} \sum_\nu \nu^\beta \left| D_\tau^{\beta-\nu} [\rho(t) D^\nu C_{1\phi}\, e^{C_{2\phi}[(\tau)]^2 \cos\phi + u^2 \cos\phi - 2\pi u)]} \right|$$

$$C_{1\phi} = \left[(2\pi\, i \sin\phi)^{-\frac{1}{2}}\right]_i e^{i\frac{\phi}{2}} \text{ and } C_{2\phi} = \frac{1}{2\, sin\, \phi}$$

$$\leq C_1\, C_{1\phi}\, e^{\left\{C_{2\phi}((a+\varepsilon)^2+|I_m u|^2 \cos\phi - 2(a+\varepsilon)|I_m u|\right\}} \max_{|\beta|\leq K} \sum_{\nu} \left| \binom{\beta}{\nu} C_{2\phi}\, \{2(a+\varepsilon)\, cos\phi \text{ -}2u\} \right|^{\nu}$$

$$\leq C_1 C_{1\phi} C_{2\phi}^k (1 + 2\, \{a \cos\phi + |u|\})^K\, e^{\left\{C_{2\phi}((a+\varepsilon)^2+|I_m u|^2) \cos\phi - 2(a+\varepsilon)|I_m u|\right\}}$$

Further proceeding as in above, it is clear that –

$$\left| \left[\left(D_{\downarrow} u^{\uparrow} m\, L\right)\right] \right|^{\uparrow} \propto (u) \,|{=}|\, \langle f(t) \left[(, D)\right]_{\downarrow} u^{\uparrow} m \left| K_{\downarrow} \phi\, (t, u) \rangle \right| \,\Big|$$

$$\leq C_m C_{1\phi} C_{2\phi}^k (1 + 2\, \{a \cos\phi + |u|\})^K\, e^{\left\{C_{2\phi}((a+\varepsilon)^2+|I_m u|^2) \cos\phi - 2(a+\varepsilon)|I_m u|\right\}}$$

Hence $L^{\propto}\,(u) \in \theta_m$ for all $u \in R^n$

θ_m is the space of all multiples of $S'\,(R^n)$.

CONCLUSION

In this paper brief introduction of generalized fractional Laplace transform is given and the analyticity theorem is proved. The fractional Laplace transform is useful in solving differential equations occurring in the branch of engineering.

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