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AN EVALUATION OF COLLECTIVE MODE Ω IN WEAK AND STRONG COUPLING LIMIT OF ONE-DIMENSIONAL FERMI GAS

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ABSTRACT

In this paper, we have studied the one-dimensional Fermi gas with attractive δ interaction potential in the quasi-particle random phase approximation at zero temperature. Using the theoretical formalism of Alm and Schuck, we have studied the behavior of collective modes over the whole coupling regime from weak coupling (high density) to strong coupling (low density). We observe that in the weak-coupling limit particle-hole RPA is approached to low momenta and the collective mode in the strong coupling limit reproduces the Bogoliubov mode for the weakly interacting bosons.

Key words: Collective mode, Quasi particle, Random phase approximation particle-hole RAP, Bogoliubov mode.

INTRODUCTION

The one dimensional Fermi gas with attractive δ interaction among the fermions has been used as a model to know the physical properties of realistic Fermi system. The exact solution for its ground state energy is known from the Bathe ansatz[1], therefore one can test approximate solutions of the systems. Quick, Esbagg and deLalno[2] have developed mean field approximation basically of two types (a) plane wave Hartre-Fock (HF) (b) non-plane wave (HF) and BCS. It was found that the BCS solution can describe the cross-over between weak coupling (as a weakly interacting gas of fermions) and strong coupling (gas of bosonic two particle pairs) in the system. In particular, they find that the BCS solution for the ground state energy coincides with the exact solution in both weak and strong coupling. In this sense, the system may serve as a simple model to study the transition between weak and strong coupling superconductivity in 1D Fermi system. This transition between weak and strong coupling has been discovered by Leggett[3] and by Noziers and Schmitt Rink[4] in three

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dimensions. The simple form of the interaction allows one to carry out approximation beyond the mean-field level such as ordinary random phase approximation (RPA's) or generalized RPA's in controlled fashion. In this approximation, one is able to calculate contribution to the ground state energy of the system beyond the mean field ground state energy. Williams and Bloch[5] have discussed the ordinary (particle hole) RPA for the one dimensional electron gas. The same approach was also applied by Bernner and Haug[6]. Friesen and Bergersen[7] have applied Singwi Sjolander generalization of RPA[8] to the ID electron gas. This model can serve as an initial approximation to realistic systems such as quasi one dimensional metal[9].

In this paper, using the theoretical formalism of T. Alm and P. Schuck[10], we have studied the collective modes of the one dimensional Fermi gas within the quasi particle random phase approximation (RPA). One calculates the collective excitations for the 1D attractive Fermi gas with a δ interaction at T = 0 by applying the quasi particle RPA. One obtains quasi particle RPA equation for the two quasi particle propagator starting from the Bogoliubov transformed Hamiltonian of the system using the equation of motion for the two particle Green's function. The homogeneous two particle equation yields the condition for the collective excitation in the system. It coincides with result found by Anderson[11], Rickayzen[12], Bardasis and Schrieffer[13] and others[14,15] by the equation of motion method.

Mathematical formulae used in the study

One writes the Hamiltonian for one dimensional Fermi gas in second quantization.

$$H = \sum_{k} \in k \; C_{\bar{k}}^{+} C_{\bar{k}} + \frac{1}{4} \sum_{k_1, k_2, k_3, k_4} \langle k_1 k_2 | \bar{V} | K_3 K_4 \rangle C_{k_1}^{+} C_{k_2}^{+} C_{k_3} C_{k_4} \quad \ldots(1)$$

where k_i denote momentum and spin quantum number of the particle and $\langle k_1 k_2 | \bar{V} | K_3 K_4 \rangle$ is the anti-symmetrized matrix element of the two body interaction. Using Bogoliubov transformation of the creation and annihilation operators Hamiltonian (1) is transformed into a new form. The transformed Hamiltonian is derived by several author[16-18], The transformed Hamiltonian is written in the following form

$$H = H^{o} + \sum_{k_1 k_2} H_{k_1 k_2}^{11} \alpha_{k_1}^{+} \alpha_{k_2} + \frac{1}{2} \sum_{k_1 k_2} \left(H_{k_1 k_2}^{20} \alpha_{k_1}^{+} \alpha_{k2}^{+} + H.C. \right) + \sum_{k_1 k_2 k_3 k_4} \left(H_{k_1 k_2 k_3 k_4}^{40} \alpha_{k_1}^{+} \alpha_{k2}^{+} \alpha_{k_3}^{+} \alpha_{k_4}^{+} + H.C. \right)$$

$$+ \sum_{k_1 k_2 k_3 k_4} \left(H_{k_1 k_2 k_3 k_4}^{31} \alpha_{k_1}^{+} \alpha_{k2}^{+} \alpha_{k_3}^{+} \alpha_{k_4}^{+} + H.C. \right) + \frac{1}{4} \sum_{k_1 k_2 k_3 k_4} H_{k_1 k_2 k_3 k_4}^{22} \alpha_{k_1}^{+} \alpha_{k2}^{+} \alpha_{k_3}^{+} \alpha_{k_4}^{+} \quad \ldots(2)$$

In equation (2), H^{o} is the BCS ground state energy and H^{11} is the diagonal part, H^{20}

is the off-diagonal part of the Hamiltonian. If one demand that the off-diagonal part H^{20} vanishes then one obtains the well known relations of BCS gap equation[19].

The H^{31} term in the Hamiltonian in eq^n (2) does not contribute to the RPA equations. The other terms H^{40} and H^{22} in the Hamiltonian describe the residual interaction among the quasi particle.

These terms are neglected in the model BCS approximation. Therefore in order to go beyond the BCS mean field approximation one has to include the residual interaction among the quasi particle.

Now, one can treat the quasi particle within the generalized RPA approximation. For this, one introduces two particles Green's function with respect to quasi particle basis and derive the equation of motion for the Green's function G. The equation has the form of Dyson equation[20] for the two particle propagator matrix G. The static elements of the mass operator A and B are given by the double commutation[21].

$$\widetilde{A}_{k_1k_2k_3k_4} = A_{k_1k_2k_3k_4} - \left(E_{k_1} + E_{k_2}\,\delta_{k_1k_2}\delta_{k_3k_4}\right)$$

$$A_{k_1k_2k_3k_4} = \left\langle\left[\alpha_{k_1}\alpha_{k_2}, \left[H_j, \alpha^{+}_{k_3}\alpha^{+}_{k_3}\right]\right]\right\rangle \qquad \ldots(3)$$

and

$$B_{k_1k_2k_3k_4} = -\left\langle\left[\alpha_{k_2}\alpha_{k_1}\left[H, \alpha_{k_1}\alpha_{k_3}\right]\right]\right\rangle \qquad \ldots(4)$$

Here $\langle\ \rangle$ refers to averaging with respect to the BCS ground state. But in place of BCS ground state one has correlated ground state which corresponds to a generalized quasi particle RPA[22].

Taking δ interaction one solves the system equation in the ω representation. One obtains coupling equation for G^{11} and G^{21} also for G^{22} and G^{21}. One obtains,

$$G_0^{11}(kq,\omega) = \frac{1}{(\omega - E_{k,q})}$$

$$G_0^{22}\ kq,\omega\ = \frac{1}{(\omega + E_{k,q})} \qquad \ldots(5)$$

Where

$E_{k,q} = E_k + E_{k+q}$

We have assumed that collective pairs have zero total spin. One arrives on the expressions

$$G^{11}(kk'q,\omega)+G^{21}(kk'q,\omega)$$

$$=G_0^{11}(kq,\omega)\delta_{k,k'}+\frac{1}{\omega^2-E_{k,q}^2}[2E_{k,q}\,m(k,q)]\,Z_{k'q,\omega}+E_{k,q}\,n(k,q)\,\Lambda_{k'q,\omega}+\omega\ell(k,q)\,\Gamma_{k'q,\omega} \quad \ldots(6)$$

and

$$G^{11}(kk'q,\omega)+G^{21}(kk'q,\omega)$$

$$=G_0^{11}(kq,\omega)\delta_{k,k'}+\frac{1}{\omega^2-E_{k,q}^2}[2\omega m(k,q)]\,Z_{k'q,\omega}+\omega n(k,q)\,\Lambda_{k'q,\omega}+\omega\ell(k,q)\,\Gamma_{k'q,\omega} \quad \ldots(7)$$

The quantities Z, Λ and Γ are given by the equations -

$$Z_{k'q,\omega}=-\frac{v}{2}\sum_{k''}m(k''q)[G^{11}(k''kq,\omega)+G^{21}(k''kq),\omega] \quad \ldots(8a)$$

$$\Lambda_{k'q,\omega}=-v\sum_{k''}n\,(k''q)[G^{11}(k''kq,\omega)+G^{21}(k''kq),\omega] \quad \ldots(8b)$$

$$\Gamma_{k'q,\omega}=-v\sum_{k''}\ell\,(k''q)[G^{11}(k''kq,\omega)+G^{21}(k''kq),\omega] \quad \ldots(8c)$$

The quantities *m*(*k*,*q*), *n*(*k*,*q*) and *l*(*k*,*q*) are combination of u_k and v_k and is given by -

$$m(k,q)=u_k v_{k+q}+v_k u_{k+q} \quad \ldots(9a)$$

$$n(k,q)=u_k v_{k+q}+v_k u_{k+q} \quad \ldots(9b)$$

$$\ell(k,q)=u_k v_{k+q}-v_k u_{k+q} \quad \ldots(9c)$$

and

$$u_k^2=1-v_k^2=\frac{1}{2}\left[1+\frac{\varepsilon_k}{E_k}\right] \quad \ldots(9d)$$

Where

$$\varepsilon_k=\in_k+v_k^{HF}-\mu$$

$$E_k=\sqrt{\varepsilon_k^2+\Delta_k^2} \quad \ldots(9e)$$

Multiplying in equation (7) subsequently with m with n and in equation (7) with ℓ and summing over k, one arrives at the system of equation for the quantities Z, Λ and Γ

$$\begin{pmatrix} 1+vI_{E,n,n}(q,\omega) & vI_{\omega,n,\ell}(q,\omega) & vI_{E,n,m}(q,\omega) \\ vI_{\omega,n,\ell}(q,\omega) & 1+vI_{E,\ell,\ell}(q,\omega) & 2vI_{\omega,\ell,m}(q,\omega) \\ \frac{v}{2}I_{E,n,m}(q,\omega) & \frac{v}{2}I_{\omega,n,\ell}(q,\omega) & 1+vI_{E,m,m}(q,\omega) \end{pmatrix}$$

$$\times \begin{pmatrix} \Lambda \\ \Gamma \\ Z_{\ell,q,\omega} \end{pmatrix} = \begin{pmatrix} -vn(k,q)G_0^{11} \\ -v\ell(k,q)G_0^{11} \\ -v/2m(k,q)G_0^{11} \end{pmatrix} \qquad \ldots(10)$$

The quantity $I_{a\,b\,c}$ are in the notation

$$I_{a,b,c} = \sum_k \frac{a(k,q)b(k,q)c(k,q)}{(\omega^2 - E_{k,q}^2)} \qquad \ldots(11)$$

Here $a(k,q) = [E_{k,q}\omega]$ and $b(k,q)$, $c(k,q) = [n(k,q), \ell(k,q), m(k,q)]$. This is a linear in homogenous system of equation for the quantities Λ, Γ and Z. This can be solved with the help of matrix inversion. Equation (10) is a eigen value problem for the determination of the collective modes in the quasi particle RPA. The condition for the nontrivial solution is the vanishing of determinant.

$$\begin{vmatrix} 1+vI_{E,n,n}(q,\Omega) & vI_{\Omega,n,\ell}(q,\Omega) & vI_{E,n,m}(q,\Omega) \\ vI_{\Omega,n,\ell}(q,\Omega) & 1+vI_{E,\ell,\ell}(q,\Omega) & 2vI_{\Omega,\ell,m}(q,\Omega) \\ v/2\,I_{E,n,m}(q,\Omega) & v/2\,I_{\Omega,n,\ell}(q,\Omega) & 1+vI_{E,m,m}(q,\Omega) \end{vmatrix} = 0 \qquad \ldots(12)$$

where Ω (q) denotes the eigen value for the collective excitation.

Weak coupling case

The analysis of the weak coupling case has been given by Belkker and Randeria[23]. One quotes their result for the weak coupling collective made in one dimension.

$$\Omega(q) = \left[1 - v\frac{1}{\pi}\frac{m}{k_F}\right]^{1/2} \frac{k_F}{m} q \qquad \ldots(13)$$

Where the density of states in one dimension for parabolic dispersion was used. The long wavelength collective modes in the weak coupling case have a phonon like spectrum and are independent of the gap. If one compares equation (13) with the small-q expansion of the collective modes in the particle hole RPA, one finds that both coincides. This means that in weak coupling the behavior of collective modes for small q is not changed from the normal particle hole RPA.

However, for large q and in particular near the point q = $2k_F$ the quasi-particle differs from the particle hole RPA.

Strong coupling limit

Now, from equation (12) one can expand the determinant of the collective modes for small -q and - Ω limit with respect to small gap. In the strong coupling limit the gap goes to zero with density.[2] One expands the gap equation in terms of Δ^2

$$1 = \frac{v}{2\pi}\int_0^\infty dk \frac{1}{|\xi_k|\left(1+\frac{\Delta^2}{|\xi_k|^2}\right)^{1/2}}$$

$$= \frac{v}{2\pi}\int_0^\infty dk \frac{1}{|\xi_k|} - v\frac{\Delta^2}{4\pi}\int_0^\infty dk \frac{1}{|\xi_k|^3} = \frac{v}{2\pi}J_1 - \frac{v\Delta^2}{4\pi}J_3 \quad \ldots(14)$$

Here one has introduced integral J_i. It has been evaluated for the strong coupling limit i.e. for $\mu^* < O$. Here μ^* is effective chemical potential including the quasi particle shift

$$J_i = \int_0^\infty dk = \frac{1}{\left(\frac{k^2}{2m}+|\mu^*|\right)^i} \quad \ldots(15a)$$

$$J_{i2} = \int_0^\infty dk = \frac{1}{\left(\frac{k^2}{2m}+|\mu^*|\right)^i} \quad \ldots(15b)$$

Then

$$J_1 = \pi m^{1/2} / \left(2|\mu^*|\right)^{1/2}$$

$$J_2 = m^{1/2}\pi / \left(2\left|\mu^*\right|\right)^{3/2}$$

$$J_3 = 3m^{1/2}\pi / \left(2^{7/2}\left|\mu^*\right|\right)^{5/2} \qquad \ldots(15c)$$

Now in equation (14) the integral in the first line is convergent for a contact interaction without a cutoff due to the one dimensionality of the system. An analogous expansion of the BCS density equation yields.

$$n = \frac{1}{2\pi}\Lambda^2 J_2 \qquad \ldots(16)$$

Now, the effective chemical potential can be expressed in terms of the density and the coupling strength as -

$$(\mu^*)^{1/2} = \frac{\sqrt{2}}{8} m^{1/2} v \pm \sqrt{\frac{2}{64} m v^2 - \frac{3}{8} n v} \qquad \ldots(17)$$

In the limit of zero density or zero gap, respectively equation (17) yields the condition.

$$2\left|\mu^*\right| = \frac{m v^2}{4} = E_0 \qquad \ldots(18)$$

This shows that in the extreme strong coupling limit the chemical potential i.e. the energy to remove a particle from the system in just half the two particle binding energy -E_O in the Vacuum.[2]

The long wavelength dispersion relation in the strong coupling limit is given by

$$\Omega(q) = cq \qquad \ldots(19a)$$

where the sound velocity c depends on the gap Δ. Now using the low density expansion of the BCS density (16) to substitute Δ^2 by the density n, one obtain the expression for the collective modes in the strong coupling limit.

$$\Omega(q) = \left[\frac{vn}{4m}\right]^{1/2} q \qquad \ldots(19b)$$

Introducing the pair mass m_B = 2m and the pair density n_B = n/2 the above expression reduces to

$$\Omega(q) = \left[\frac{vn_B}{4m}\right]^{1/2} q \qquad \ldots(19c)$$

Equation 19(c) is the well known Bogoliubov dispersion relation for the weakly interacting Bose gas[24,25] in the limit of small q which is linear in q i.e. phonon like. Thus starting from interacting fermions with an attractive interaction the quasi particle RPA in the strong coupling limit yields the dispersion relation for weakly interesting gas of bosons (two particle bound states). The magnitude of the repulsive interaction among the bosons in equation 19(c) is given by the fermionic interaction strength v and is consistent with the result of Hanssmann[26] and others.[27-32]

RESULTS AND DISCUSSION

In this paper, using the theoretical formalism of T. Alm and P. Schuck[10], we have studied the behavior of collective modes over the coupling regime weak coupling (high density) to strong coupling (low density). We observe that the treatment of the residual interaction in the Hamiltonian (2) within the quasi particle or generalized RPA allows one to study the behavior of the collective over the whole coupling range. Collective modes in one dimension were found for the case of an attractive δ interaction. One observes that in the weak coupling limit one recovers Anderson's results[11] whereas in the strong coupling limit the Bogoliubov dispersion relation[24] for the interacting Bose gas of two particle pairs can be derived from the quasi particle RPA. This is consistent with the fact that the BCS theory is capable of describing the extreme strong coupling limit i.e. the gas of two particle bound state properly and reproduces the exact result for the ground state energy in this limit.[2] In Table 1 we have presented for small momenta q in the weak coupling limit ($\mu^* = 3.386\ E_o$).

Table 1: An Evauated results of collective mode Ω for small momenta q in the weak coupling limit ($\mu^*= 3.386\ E_0$) the numerical solution is compared with weak coupling expansion (13).

q/k_F	Ω/μ	
	Numerical solution equation (12)	**Weak coupling expansion eq. (13)**
0	0.00	0.00
0.1	0.126	0.132
0.2	0.268	0.275
0.3	0.355	0.368

Cont…

q/k_F	Ω/μ	
	Numerical solution equation (12)	**Weak coupling expansion eq. (13)**
0.4	0.568	0.599
0.5	0.573	0.702
0.6	0.626	0.925
0.65	0.645	1.027
0.70	0.689	1.146
0.75	0.735	1.207
0.80	0.794	1.258

The numerical solution equation (12) is compared with weak coupling expansion equation (13). Both solution coincides at q/kF = 0.4. This shows the consistency of the numerical solution with the well known weak coupling result which was obtained by Anderson[11] in the 3D case. In Table 2, we have presented the evaluated results of collective mode Ω for small momentum q in the strong coupling case (μ^* = (-) 0.4966 E_0)' The numerical solution equation (12) is compared with the strong coupling expansion (equation 19c) and also with free particle dispersion $\Omega = q^2/2m_B$. We observe that the numerical solution starts linearly in q and is consistent with the strong coupling expansion gives in equation 19(c). This confirms the interpretation of the collective excitations in the strong coupling limit as Bogoliubov sound modes of the two particle Bose gas that is formed in the limit. The free particle solution reached the full solution for large q.

Table 2: An Evauated results of collective mode Ω for small momenta q in the strong coupling limit (μ^*= (-) 0.4966 E_0) The numerical solution [eq. (12)] is compared to the strong coupling expansion [eq. (19c)] and free particle dispersion $\Omega = q^2/2m_B$ (a_0 = Bohr radius)

q/a_0	Ω/E_o		
	Numerical solution eq. (12)	**Strong expansion eq. (19c)**	**Free particle dispersion $\Omega = q^2/2m_B$**
0	0.000	0.000	0.000
0.02	0.0022	0.0020	0.0002
0.04	0.0032	0.0032	0.0004

Cont…

q/a_0	Ω/E_o		
	Numerical solution eq. (12)	**Strong expansion eq. (19c)**	**Free particle dispersion $\Omega = q^2/2m_B$**
0.05	0.0034	0.0035	0.0005
0.06	0.0037	0.00355	0.00282
0.07	0.0038	0.00362	0.00294
0.08	0.0042	0.00375	0.00307
0.09	0.0048	0.00382	0.00322
0.10	0.0052	0.00397	0.00375
0.11	0.0056	0.00416	0.00400
0.12	0.0062	0.00438	0.00425
0.15	0.0094	0.00469	0.00473
0.20	0.0126	0.00525	0.00098

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