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About the two solutions of the Ising model

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ABSTRACT

Previously, the author found the exact solutions of the one-dimensional, two-dimensional, and three-dimensional Ising models. Those solutions produced two solutions: one with a spontaneous magnetisation and one with zero total magnetisation. In the present paper it is shown that the latter is unlikely. Other evidence is given which supports the previous results of the author.

KEYWORDS

Ising model;
Magnetic materials;
Nanostructures;
Magnetic properties;
Magnetic structure.

THEORY

In Reference 1, the exact solutions of the one-dimensional, two-dimensional, and three-dimensional Ising models were obtained. They are as follows. The magnetisation of the linear chain is:

$$\sigma = [\tanh(\beta J)]^{0.5} \quad (1)$$

Here J is the interaction energy between nearest neighbour spins and $\beta = \frac{1}{kT}$ [1,2] The magnetisation of the square lattice is:

$$\sigma = \left[\frac{\sinh(2\beta J)}{\cosh(2\beta J)+3}\right]^{0.5} \quad (2)$$

The magnetisation of the hexagonal lattice is:

$$\sigma = \left[\frac{\exp(1.5\beta J)-\exp(-0.5\beta J)}{\exp(1.5\beta J)+3\exp(-0.5\beta J)}\right]^{0.5} \quad (3)$$

And the magnetisation of the simple cubic lattice is:

$$\sigma = \left[\frac{\sinh(3\beta J)+4\sinh(1.5\beta J)+5\sinh(\beta J)+4\sinh(0.5\beta J)}{\cosh(3\beta J)+8\cosh(1.5\beta J)+15\cosh(\beta J)+24\cosh(0.5\beta J)+16}\right]^{0.5} \quad (4)$$

These solutions display spontaneous magnetisation which acquires its maximum value at zero temperature and decreases asymptotically to zero with temperature.

However, the exact solution produces another solution, namely, the one with zero magnetisation for all dimensions. In the one-dimensional case with the nearest-neighbour correlation function $\Gamma_k(1) = 0.99$ (here k is the number of a spin and 1 is the distance between spins, in units of the lattice constant)[1], the chain consists of domains, each consisting of an average of about 50 spins. In some of the domains the spins are turned up, and in the others they are turned down. The net magnetisation is zero. This is also a magnetic chain, so-called domain magnetism. In a non-magnetic chain, according to the definition, there are equal numbers of up and down spins, and they are oriented randomly; that is, its correlation $\Gamma_k(1) = 0$. Let us decrease the temperature T; then the correlation function will increase. When $\Gamma_k(1) = 0.999$ then the chain consists of domains of about 500 spins; when $\Gamma_k(1) = 0.9999$ then the domains have about 5000 spins, and so on. At $T = 0$, $\Gamma_k(1) = 1$, all of the spins are up and therefore the spins

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in the domains where they were down have flipped. This is a discontinuity in the Ising solution, and hence such a solution is unlikely. One can easily see that this conclusion is valid for higher dimensions. Earlier it was supposed by the author that the non-zero solution is correct because of the symmetry of the Hamiltonian[1]. The solution with non-zero magnetism is more symmetric than that with the zero total magnetism.

In Reference 1 it was proved that the magnetisation of the Ising chain which consists of only two spins is given by Equation 1, regardless of whether that chain is independent or embedded into a longer chain. The magnetisation in Equation 1 does not depend on the number of spins in the chain. Therefore, the total magnetisation of a long Ising chain is equal to the sum of the magnetisations of its links (elementary cells) which consist of two spins. This leads to an important result: for a uniform chain (J for all spins is the same), the long range order of spins equals the short range order and it is enough to define the total magnetisation of the chain by the magnetisation of its elementary cell. For example, when $\Gamma_k(1) = 0.99$, the probability that the neighbouring spin in a cell points in the same direction is 99%. For all other cells that probability is the same.

Let us consider Equations 11 and 12 from Reference 1, here Equations 5 and 6, in more detail. "The magnetisation of an Ising chain is equal to that of an averaged elementary cell."[1] The average square magnetisation of a chain is

$$\sigma^2 = \frac{\sum_{i=1}^{N-1} \tanh(\beta J_i)}{N-1} \quad (5)$$

where N is the number of spins. In the uniform case, $J_i = J$,

$$\sigma^2 = \tanh(\beta J) \quad (6)$$

which results in Equation 1. However, in a non-uniform case, instead of Equation 5 it is better to use Equation 7:

$$\sigma = \frac{\sum_{i=1}^{N-1} [\tanh(\beta J_i)]^{0.5}}{N-1} \quad (7)$$

because it is more accurate.

Let us consider a uniform two-dimensional Ising model on a square lattice. Let us suppose that the traditional "exact" solution is correct. At $T = 0$, $\Gamma_k(1) = 1$ and all spins are up. At a small temperature close to 0, $\Gamma_k(1)$ will be equal to 0.99 for all k (that is, for all spins) in both directions because the problem is symmetric. That means that mathematically, two solutions are possible. In the first, the lattice consists of rectangular domains with both sides having about 50 spins; in some domains the spins are up and in the others, down, and the total magnetisation is zero. This solution contradicts the traditional one. The second solution is where about 99% of the spins are up in both directions. That means that the magnetisation of the lattice is determined by the magnetisation of one cell, the long range order of spins equals the short range one (because $\Gamma_k(1) = 0.99$ for all k), and the equation of magnetisation of one cell is Equation 2 but not the traditional one. As the magnetisation is described by one and the same equation, Equation 2 will also be valid for other values of $\Gamma_k(1)$: 0.5, 0.4, …, and so on. The same reasoning is valid for three dimensions. For $T = 0$, $\Gamma_k(1) = 1$. At a temperature close to 0, $\Gamma_k(1)$ will equal 0.99 for all k in all three directions; therefore, the long range order of spins is equal to the short range order and the magnetisation of the lattice is described by that of one cell by Equation 4.

The outcome that the long range order of spins in the Ising model equals the short range one is of great importance. It allows factorisation of the partition function of the Ising model in two and three dimensions. The partition function of the two- or three-dimensional lattice in the uniform case is:

$$Z = \sum_{\{s\}} \exp[\beta(E_1 + E_2 + E_3 + \ldots)] = Z_1 Z_2 Z_3 \ldots \quad (8)$$

where E_i and Z_i are the energies and the partition functions of the elementary cells, and the symbol $\{s\}$ denotes the N-fold summation[1].

CONCLUSIONS

It has been shown that the solutions of the one-dimensional, two-dimensional, and three-dimensional Ising models with zero magnetisation are less likely than the solutions with a spontaneous magnetisation for all temperatures. The former exhibit discontinuity and flipping of spins at zero temperature. Other proofs of incorrectness of the previous "exact" solutions of the one-dimensional and two-dimensional models and of the correctness of the solutions given in Reference 1 are provided.

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REFERENCES

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