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A STUDY OF CONTINUUM THEORY OF TKACHENKO MODES IN ROTATING BOSE-EINSTEIN CONDENSATE

JAYA PRAKASH SINHA[*] and L. K. MISHRA[a]

Department of Physics, Shyam Nandan Sahaya College, MUZAFFARPUR – 843116 (Bihar) INDIA
[a]Department of Physics, Magadh University, BODH-GAYA – 824234 (Bihar) INDIA

ABSTRACT

Tkachenko mode is a transverse sound wave in a vortex lattice, which exists in the ground state of a rotating super fluid. It has attracted a lot of attention in vortex dynamics of laboratory and astrophysical super fluids. Observation of this extremely soft mode (of order or less than a few Hz) in laboratory super fluids was very difficult because even slight pinning of vortex ends the Tkachenko wave into a classical inertial wave. This mode is a direct manifestation of the quantum vortices since it depends on the circulation quantum. Tkachenko mode in rotating BEC was investigated numerically with solving the equations of Gross-Pitaevskii theory, but the numerical results do not agree with the experiment.

In the present paper, it has been shown that the continuum theory of Tkachenko modes, taking into account density, inhomogenity and compressibility of the condensate, successfully explained the observed experimental data.

Key words: Tkachenko mode, Super fluid, Bose-Einstein condensate, Circulation quantum, Gross-Pitaevskii equation.

INTRODUCTION

Tkachenko mode[1] is a transverse sound wave in a vortex lattice, which exists in the ground state of a rotating super fluid. It has attracted a lot of attention in vortex dynamics of vortex dynamics of laboratory and astrophysical superfluids[2]. Observation of this extremely soft mode (of order or less than a few Hz) in laboratory super fluids was very difficult because even slight pinning of vortex ends shades the contribution of vortex shear rigidity transforming the Tkachenko wave into a classical inertial wave[3]. However, discovery of the Bose-Einstein condensate (BEC) and the possibility to rapidly rotate it made observation of

[*]Author for correspondence; E-mail: muphysicslkm@gmail.com

Tkachenko modes more feasible, which resulted in clear experimental detection of them[4]. As well as the Kelvin mode, also possible in the BEC[5], the Tkachenko mode is a direct manifestation of the quantum vortices since it depends on the circulation quantum.

In a number of aspects, external conditions for Tkachenko waves in BEC are essentially different from those in 'old' super fluid. First, in a rapidly rotating BEC, one cannot consider a super fluid to be incompressible. The effect of finite compressibility was investigated theoretically in the past[3], since the effect is crucial for the ling-wavelength limit of the Tkachenko mode. But at that time, it was considered as a theoretical curiosity since in order to reveal it, one needed containers about a few hundred meters in diameter. In the BEC case, the situation is essentially different because in contrast to a strongly interacting Bose liquid such as He II, BEC is a weakly interacting Bose gas with very low sound speed and very high compressibility. The importance of high liquid compressibility for Tkachenko wave in BEC was pointed out by Baym[6], who red rived the spectrum of Tkachenko waves in a compressible liquid[7] known from Ref. 3 and compared it with the experiment[4]. Another importance feature of rotating BEC, also connected with its high compressibility, is that the liquid density is essentially inhomogeneous. This feature was taken into account in the theory by Anglin and Crescimahno[8], which the frame of the continuum theory, which replaces a discrete vortex lattice by a continuous medium, like the elasticity theory for atomic crystals. But they neglected liquid compressibility, while a proper comparison with the experiment requires a theory taking into account both features, compressibility and in homogeneity. Recently, the Tkachenko mode in rotating BEC was investigated numerically with solving the equations of Gross-Pitaevskii theory (mean-field theory)[9,10]. The numerical results agreed well with the experiment. It would be also useful to develop an analytical approach since it could provide a deeper insight into physics of the phenomenon. In the experiment, there are good conditions for application of a continuum theory since usually the relevant length scale (the condensate size) essentially exceeded the inter vortex distance. The goal of the present paper is to suggest the continuum theory of Tkachenko modes in a rapidly rotating BEC taking into account liquid compressibility and inhomogeneity. The main challenge for the theory was to formulate proper boundary conditions for oscillating BEC.

Mathematical formulae used in the evaluation

One considers equations of motion of homogeneous compressible super fluid at T = 0 (no normal component) in the rotating coordinate frame[3] -

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla . V = 0 \qquad \ldots(1)$$

$$\rho_0(\frac{\partial V}{\partial t} + 2\Omega \times V_L) = -C_s{}^2 \nabla \rho' \quad \ldots(2)$$

$$\rho_0 \kappa \times (V_L - V) = \rho_0 \frac{\kappa C_T{}^2}{2\Omega}[2\nabla(\nabla . u) - \Delta u] \quad \ldots(3)$$

Here $\rho'(\vec{r})$ is the oscillating component of the liquid mass density $\rho(\vec{r}) = \rho_o + \rho'(\vec{r})$ around the equilibrium homogeneous density ρ_0, u is the vortex displacement, $V_L = \frac{du}{dt}$ is the vortex velocity, V is the liquid velocity averaged over the vortex-lattice cell, Ω is the angular velocity, C_s is the sound velocity, κ = h/m is the circulation quantum and C_T=$C_T = (\frac{\kappa\Omega}{8\pi})^{\frac{1}{2}}$ is the Tkachenko-wave velocity. Assuming that $C_T \ll C_S$ and $C_T \kappa \ll \Omega$, one receives for the spectrum of plane wave $\propto \exp(i\kappa . r - i\omega t)$ the dispersion relation

$$\omega^4 - \omega^2(4\Omega^2 + C^2{}_s\kappa^2) + C^2{}_s C^2{}_T \kappa^4 = 0 \quad \ldots(4)$$

which yields the gaped sound mode

$$\omega^2 = 4\Omega^2 + C^2{}_s \kappa^2 \quad \ldots(5)$$

and the soft quantum mode (Ref. 11 and 13)

$$\omega^2 = \frac{C^2{}_s C_T{}^2 \kappa^4}{4\Omega^2 + C_s{}^2 \kappa^2} \quad \ldots(6)$$

So, for the Tkachenko mode, compressibility is essential in the long-wavelength limit $\kappa \ll \Omega / C_s$. It transforms the Tkachenko wave with the sound spectrum $\omega = C_T \kappa$ to a softer mode $\omega \propto \kappa^2$. In the Tkachenko mode motion of both the vortex lattice and the liquid has an elliptic polarization but longitudinal components parallel to the wave vector κ are very small. The transverse components (normal to κ) of the vortex lattice and the liquid are close to one another; $V_{Lt} = V_r$.

Generalization of these equations onto an inhomogeneous liquid with equilibrium density $\rho_0(r)$, which varies in the plane normal to the rotation axis, is straight forward. In particular, one should replace $\rho_0 \nabla . V$ in Eq. (1) by $\nabla . [\rho_0(r) V]$. The goal of this paper is to consider only axis symmetric Tkachenko eigen modes in an axis symmetric rotating BEC. Thus, one writes the equations of motion in the polar system of coordinates for the nomochromatic mode $\propto^{-i\omega t}$.

$$2\Omega i\omega V_r = -\omega^2 V_t - \frac{C_T{}^2}{\rho_0}\frac{1}{r^2}\frac{\partial}{\partial r}[\rho_0 r^3 \frac{\partial}{\partial r}(\frac{V_T}{r})] \quad \ldots(7)$$

$$2\Omega i\omega V_t = \frac{C_s{}^2}{\rho_0}\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial(\rho_0 r V_r)}{\partial r}] \quad \ldots(8)$$

One excluded all variables except for the tangential (azimuthal in polar coordinates) component of the velocity $V_t = V_{Lt}$ and the radial component of the liquid velocity V_r is crucial for the compressibility effect. For a weakly interacting Bose gas, C_s^2 is proportional to the density ρ_0. Therefore, the ratio C_s^2 / ρ_o can be replaced by its value $C_s^2(0)/\rho_0(0)$ in the BEC centre r = 0.

Let us discuss first the results for Tkachenko eigenmodes in an incompressible ($C_{s\to\infty}$) homogeneous ($\rho_0 = const.$) liquid in a cylindrical container. Then the radial liquid a velocity V_r vanishes and Eq. (7) is the Bessel equation with a solution $V_t \propto J_l(\kappa r)$. The eigen values of the wave number κ are determined from the boundary condition at r = R, where R is the radius of the vortex-lattice sample, which is normally close to the container radius. In the ideal axis symmetric case without any interaction of vortices with lateral walls the radial flux of the azimuthal component of the momentum, which is given by the corresponding component of the vortex-lattice stress tensor, must vanish. This gives the boundary condition[3].

$$\frac{du(R)}{dr} - \frac{u(R)}{R} = -\frac{1}{i\omega}[\frac{dV_t}{dr} - \frac{V_t(R)}{R}] = 0 \quad \ldots(9)$$

The flux of the angular moment into the liquid is proportional to the same stress-tensor component and therefore vanishes also. Equation (9) leads to the condition $J_2(\kappa R) = 0$ imposed on the wave number κ. Traditionally, in the papers on BEC, they scale the Tkachenko mode frequencies by the frequency $\Omega b / R$, where $b = \sqrt{\kappa / \sqrt{3\Omega}}$ is the inter vortex distance[4,8,10].

$$\omega = \tilde{\omega}\frac{\Omega b}{R} = \tilde{\omega}\sqrt{\frac{8\pi}{\sqrt{3}}}\frac{c_T}{R} \quad \ldots(10)$$

Then the first two roots κR = 5.14 and 8.42 of the equation $J_2(\kappa R) = 0$ give the first two reduced eigen frequencies of the Tkachenko mode $\tilde{\boldsymbol{\omega}}\sqrt{\sqrt{3}/8\pi kR} = 0.263\, kR$ with the ratio 8.42/5.14 = 1.64. Another interesting case is the limit of the strong interaction of vortices with rough lateral walls with $V_t(R) = 0$. Then the eigen frequencies are given by the first two roots of the equation $J_1(\kappa R) = 0 \kappa R = 3.83$ and 7.02 with the ratio of the two lowest eigen frequencies 7.02/3.83 = 1.83.

RESULTS AND DISCUSSION

In this paper, we have studied the continuum theory of Tkachenko modes in a rotating two-dimensional Bose-Einstein condensates. We have taken the theoretical formalism of Sonies[11], who has taken into account the density, inhomogeneity and compressibility of the condensate. The theory is based on the solution of the coupled hydrodynamic equation of vortex and liquid motion with proper boundary conditions. These boundry conditions for the condensate have been obtained using Thomas-Fermi approximation[12]. We have computed the eigen frequency $\tilde{\omega}_1$ as a function of $\Omega / \sqrt{\omega_t^2 - \Omega^2} = s / 2\sqrt{2}$. The results are shown in Table 1.

Table 1: An Evaluated results of eigen frequency $\tilde{\omega}_1$ as a function of $s / 2\sqrt{2}$. where $s = \frac{2\sqrt{2}\Omega}{\sqrt{(\omega_\perp^2 - \Omega^2}}$. Ω is the angular velocity of rotation. Theoretical results were compared with the experimental datas. $\omega_\perp$ is the trap frequency, *b* is the inter vortex displacement, R is the BEC border, r = R

$s = \frac{2\sqrt{2}\Omega}{\sqrt{\omega_\perp^2 - \Omega^2}}$	$\tilde{\omega}_1 = \frac{\omega_\perp R}{\Omega b}$	
	Theo	**Expt.**
1.0	1.427	-
1.5	1.395	-
2.0	1.374	1.245
2.5	1.372	1.206
3.0	1.356	1.168
3.5	1.289	1.138
4.0	1.165	1.105
4.5	1.105	0.953
5.0	0.985	0.886
5.5	0.856	0.754
6.0	0.683	0.652
7.0	0.605	0.626

The theoretical results were compared with the experimental data. Qualitative agreement between the theory and the experiment looks quite good. However, the disagreement is well the numerical data. It becomes worse at large s. However, this disagreement is not connected with the growth of s itself, which accompanies the growth of s in accordance with the ξ/b. One believes that better agreement with larger 's' can be obtained, if the experiments are performed with larger number of atoms[13,14]. Conddington et al.[15] measured the ratio of the two first frequencies, which corresponds[15,16] to s = 8.61. The present theory predicts the ratio ω_2/ω_1 = 2.09. Some recent calculations[17-20] also reveal the same facts.

CONCLUSION

The continuum theory of Tkachenko modes was developed, which takes into account the liquid inhomogeneity and finite compressibility. When this theory was applied to rapidly rotating BEC with proper boundary conditions, the theory is found in good agreement with the observation of Tkachenko modes.

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