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A New Continued Fraction Approximation of the Gamma Function based on the Burnside's Formula

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Abstract

In this paper, based on the Burnside's formula, we establish a new continued fraction approximation and inequalities of gamma function. Finally, for demonstrating the superiority of our new approximation over several formulas, we give some numerical computations.

Keywords: Gamma function; Continued fraction; Burnside's formula

I. Introduction

Many mathematicians have made great efforts in the area of establishing more precise inequalities and accurate approximations for the factorial function and its extension of gamma function.

The gamma function is defined as follow:

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, x>0 \tag{1.1}$$

Stirling's formula,

$$n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.2}$$

is one of the most widely known formulas for approximation of the factorial function.

Based on this formula, lots of approximation formulas were discovered. The Stirling's series [1]:

$$n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\cdots\right) \tag{1.3}$$

which is the extension of (1.2).

Burnside's formula [2],

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$$n!\approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}, \tag{1.4}$$

which is more precise than (1.2). Also, there are many approximations which are better than (1.4), Dawei Lu's formula [3],

$$n!\approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\cdots\right)^{\frac{1}{k}}, \tag{1.5}$$

where

$$c_1=-\frac{k}{24},\ c_2=\frac{k^2}{1152}+\frac{k}{48},\ c_3=-\frac{23k}{2880}-\frac{k^2}{1152}-\frac{k^3}{82944}\cdots,$$

and Dawei Lu's continued fraction formula [4],

$$n!\approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\ddots}}}\right)^{\frac{n-\frac{1}{2}}{k}}, \tag{1.6}$$

where

$$a_1=-\frac{k}{24},\ a_2=\frac{k}{48}-\frac{23}{120},\ a_3=\frac{14}{5k-46}\ \cdots.$$

Also, some authors paid attention to giving the better increasing approximations of the gamma function using continued fraction. For example, Mortici introduced Stieltjes' continued fraction [5],

$$\Gamma(x+1)\approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\exp\left(\cfrac{a_0}{x+\cfrac{a_1}{x+\cfrac{a_2}{x+\cdots}}}\right), \tag{1.7}$$

Where

$$a_0=\frac{1}{12},\ a_1=\frac{1}{30},\ a_2=\frac{53}{210},\ \cdots,$$

and Mortici also provided a new continued fraction approximation as follows [6]:

$$\Gamma(x+1) \approx \sqrt{2\pi x}e^{-x}\left(x+\cfrac{1}{12x-\cfrac{1}{10x+\cfrac{a}{x+\cfrac{b}{x+\cfrac{c}{x+\cfrac{d}{x+\ddots}}}}}}\right)^{x}, \tag{1.8}$$

Where

$$a=-\frac{2369}{252}, b=\frac{2117009}{1193976}, c=\frac{393032191511}{1324011300744}, d=\frac{33265896164277124002451}{142780241040896418788840}\cdots.$$

ChaoPing Chen provided a new approximation starting from Burnside's formula (1.4) as follows [7]:

$$\Gamma(x+1) \approx \sqrt{2\pi}\left(\frac{x+\frac{1}{2}-\frac{\frac{1}{24}}{x+\frac{1}{2}}+\frac{\frac{19}{5760}}{(x+\frac{1}{2})^3}-\frac{\frac{2561}{29030401}}{(x+\frac{1}{2})^5}+\cdots}{e}\right)^{x+\frac{1}{2}} \tag{1.9}$$

In this paper, we provide a new continued fraction approximation of the gamma function starting from Burnside's formula (1.4) as follows:

Theorem 1.1.

For the factorial function, we have,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\cfrac{1}{a_1n+b_1+\cfrac{1}{a_2n+b_2+\cfrac{1}{\ddots+\cfrac{1}{a_in+b_i}}}}\right), \tag{1.10}$$

Where

$$a_1=-24, b_1=-12, a_2=-\frac{5}{7}, b_2=-\frac{5}{14}, a_3=-\frac{8232}{1517}, b_3=-\frac{4116}{1517} \quad \cdots.$$

Next, using Theorem 1.1, we provide some inequalities for the gamma function.

Theorem 1.2.

For every $x \geq 0$, it holds:

$$\exp\left(\frac{1}{-24x-12}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}} < \exp\left(\frac{1}{-24x-12-\frac{1}{\frac{5}{7}x+\frac{5}{14}}}\right) \tag{1.11}$$

To prove Theorem 1.1, we need the following lemma which is very useful for constructing asymptotic expansions and for accelerating some convergences.

Lemma 1.1.

If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit,

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in (-\infty,+\infty) \tag{1.12}$$

with s>1, then,

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1} \tag{1.13}$$

Lemma 1.1 was first proved by Mortici in [8]. Using Lemma 1.1, we can see that the rate of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value *s* satisfying (1.12).

The rest of this paper is arranged as follows. In Section II, we provide the proof of Theorem 1.1. In Section III, the proof of Theorem 1.2 is given. In Section 4, we give some numerical computations which demonstrate the superiority of our new approximation over the Burnside formula, the Dawei Lu's formula and the ChaoPing Chen's formula.

II. Proof of Theorem 1.1

First, we need to find the value of the parameters a_1, $b_1 \in \mathbb{R}$ which produces the best approximation of the form,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{1}{a_1 n + b_1}\right). \tag{2.1}$$

A method to measure the accuracy of approximation (2.1), is to define the sequence $(\omega_n)_{n\geq 1}$ by the relation,

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{1}{a_1 n + b_1}\right)\exp(\omega_n) \tag{2.2}$$

and to say that an approximation (2.1) is better if $(\omega_n)_{n\geq 1}$ converges to zero faster.

From (2.2), we have,

$$\omega_n = \ln n! - \frac{1}{2}\ln 2\pi - \left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right) + n + \frac{1}{2} - \frac{1}{a_1 n + b_1}. \tag{2.3}$$

Thus,

$$\omega_n-\omega_{n+1}=-1+\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n+\frac{1}{2}}\right)+\ln\left(1+\frac{\frac{1}{2}}{n+1}\right)-\frac{1}{a_1n+b_1}+\frac{1}{a_1(n+1)+b_1} \tag{2.4}$$

Developing (2.4) in a power series in $\frac{1}{n}$, we have,

$$\omega_n-\omega_{n+1}=\left(-\frac{1}{24}-\frac{1}{a_1}\right)\frac{1}{n^2}+\left(\frac{1}{12}+\frac{1}{a_1}+\frac{2b_1}{a_1^{\ 2}}\right)\frac{1}{n^3}+\left(-\frac{41}{320}-\frac{1}{a_1}-\frac{3b_1}{a_1^{\ 2}}-\frac{3b_1^{\ 2}}{a_1^{\ 3}}\right)\frac{1}{n^4}+O(\frac{1}{n^5}) \tag{2.5}$$

Thus, using Lemma 1.1, we have,

- If $a_1 \neq -24$, then the rate of convergence of $(\omega_n)_{n\geq 1}$ is n^{-1} , since,

$$\lim_{n\to\infty} n\omega_n = -\left(\frac{1}{24}+\frac{1}{a_1}\right)\neq 0\cdot$$

- If $a_1 = 24, b_1 \neq -12$, then from (2.5), we have,

$$\omega_n-\omega_{n+1}=\left(\frac{1}{24}+\frac{b_1}{288}\right)\frac{1}{n^3}+O(\frac{1}{n^4})$$

  and the rate of convergence of $(\omega_n)_{n\geq 1}$ is n^{-1} , since,

$$\lim_{n\to\infty} n^2\omega_n = \frac{1}{2}\left(\frac{1}{24}+\frac{b_1}{288}\right)\neq 0$$

- If $a_1 = -24, b_1 = -12$,then, from (2.5), we have,

$$\omega_n-\omega_{n+1}=\frac{7}{960n^4}+O(\frac{1}{n^5})$$

  and the rate of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-3}, since,

$$\lim_{n\to\infty} n^3\omega_n = \frac{7}{2880}$$

We can see that the fastest possible sequence $(\omega_n)_{n\geq 1}$ is obtained only for $a_1 = -24, b_1 = -12$.

Next, we define the sequence $(u_n)_{n\geq 1}$ by the relation,

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\exp\left(\frac{1}{-24n-12+\frac{1}{a_2n+b_2}}\right)\exp(u_n), \tag{2.6}$$

where a_2, b_2 is any real number. We use the same method (2.1) to (2.5). Then, we can see that the best possible sequence $(u_n)_{n\geq 1}$ is obtained only for $a_2 = -\frac{5}{7}, b_2 = -\frac{5}{14}$.

Then, we define the sequence $(v_n)_{n\geq 1}$ by the relation,

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{1}{-24n-12+\frac{1}{-\frac{5n}{7}-\frac{5}{14}+\frac{1}{a_3 n+b_3}}}\right)\exp(\upsilon_n). \tag{2.7}$$

Using the same method from (2.1) to (2.5), we can see that the fastest possible sequence $(\upsilon_n)_{n\geq 1}$ is obtained only for $a_3 = -\frac{8232}{1517}, b_3 = -\frac{4116}{1517}$.

Similarly proceeding, we get $a_i, b_i (i = 1,2, \ldots)$. For example,

$$a_4 = -\frac{2301289}{6918030}, b_4 = -\frac{2301289}{13836060},$$

$$a_5 = -\frac{286503689736}{96288327103}, b_5 = -\frac{143251844868}{96288327103},$$

$$a_6 = -\frac{52374449785240453}{241778324720548917}, b_6 = -\frac{52374449785240453}{483556649441097834}.$$

$$a_7 = -\frac{1026004184758180160947016 40}{50286103698593621472051013}, b_7 = -\frac{51300209237909008047350820}{50286103698593621472051013},$$

$$a_8 = -\frac{1067010697119586920748445005718596 2833}{66384209830428923589901518762311503292}$$

$$b_8 = -\frac{10670106971195869207484450057185962833}{132768419660857847179803037524623006584},$$

$$\ldots$$

We obtain the new asymptotic expansion (1.10).

III. Proof of Theorem 1.2

The left-hand side inequality of (1.11) is equivalent to $(x) > 0$, where,

$$F(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + x + \frac{1}{2} + \frac{1}{24x+12} \tag{3.1}$$

Let,

$$f(x) = F(x) - F(x+1).$$

From (3.1), we have,

$$f(x) = -1 + \ln\frac{1}{x+1} - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + \left(x+\frac{3}{2}\right)\ln\left(x+\frac{3}{2}\right) + \frac{1}{24x+12} - \frac{1}{24(x+1)+12}, \tag{3.2}$$

From (3.2), we have, for every ≥ 0 ,

$$f''(x) = \frac{25+56x+28x^2}{3(1+x)^2\left(3+8x+4x^2\right)^3}, \tag{3.3}$$

$f''(x) > 0$, so $f'(x)$ is strictly increasing on$[0,\infty]$. As $\lim_{x\to\infty} f'(x) = 0$, we get $f'(x) < 0$ on $[0,+\infty]$. Thus, $f(x)$is strictly decreasing on $[0,+\infty]$. As $\lim_{x\to\infty} f'(x) = 0$, we get $f(x) > 0$ on $[0,+\infty]$. So,

$$F(x) > F(x+1) > \cdots > F(x+n),\ F(x) > \lim_{n\to+\infty} F(x+n) = 0.$$

Then, the left-hand side inequality in Theorem 1.2 is obtained.

The right-hand side inequality of (1.11) is equivalent to $G(x) < 0$, where,

$$G(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + x + \frac{1}{2} + \frac{1}{24x+12+\frac{1}{\frac{5}{7}x+\frac{5}{14}}}. \tag{3.4}$$

Let,

$$g(x) = G(x) - G(x+1).$$

From (3.4), we have,

$$g(x) = -1 + \ln\frac{1}{x+1} - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + \left(x+\frac{3}{2}\right)\ln\left(x+\frac{3}{2}\right) + \frac{1}{24x+12+\frac{1}{\frac{5}{7}x+\frac{5}{14}}} - \frac{1}{24(x+1)+12+\frac{1}{\frac{5}{7}(x+1)+\frac{5}{14}}}. \tag{3.5}$$

From (3.5), we have, for every $x > 0$,

$$g''(x) = \frac{-h(x)}{(1+x)^2(3+8x+4x^2)(37+120x+120x^2)^3(277+360x+120x^2)^3}, \tag{3.6}$$

Where,

$$h(x) = 800369777449 + 7567926713280\,x + 30646991363040\,x^2 + 69282414374400\,x^3$$

$$+95510046153600\,x^4 + 82144958976000\,x^5 + 43050237696000\,x^6 + 12582604800000\,x^7$$

$$+1572825600000\,x^8.$$

$g''(x) < 0$, so $g'(x)$ is strictly decreasing on $[0,+\infty]$. As $\lim_{x\to+\infty} g'(x) = 0$, we get $g'(x) > 0$ on $[0,+\infty]$. Thus, $g(x)$ is strictly increasing on$[0,+\infty]$. As$\lim_{x\to+\infty} g(x) = 0$, we get $g(x) > 0$ on$[0,+\infty]$. So,

$$G(x) < G(x+1) < G(x+n),\ G(x) < \lim_{n\to+\infty} G(x+n) = 0.$$

Then, the right-hand side inequality in Theorem 1.2 is obtained.

IV. Numerical Computations

Here, we give a comparison table to demonstrate the superiority of our new approximation,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{1}{a_1 n+b_1+\frac{1}{a_2 n+b_2+\frac{1}{\ddots+\frac{1}{a_i n+b_i}}}}\right) = \theta_{i,n} \tag{4.1}$$

over Burnside’s formula,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \alpha_n \tag{4.2}$$

Dawei Lu's formula,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1-\frac{k}{24n}+\left(\frac{k^2}{1152}+\frac{k}{48}\right)\frac{1}{n^2}\right)^{\frac{1}{k}}=\beta_{k,n} \tag{4.3}$$

Dawei Lu's continued fraction formula,

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\frac{-\frac{k}{24}}{n^2+\frac{(\frac{k}{48}-\frac{23}{120})n}{n+\frac{14}{5k-46}}}\right)^{\frac{n-\frac{1}{2}}{k}}=\gamma_{k,n} \tag{4.4}$$

and Chao-Ping Chen's formula,

$$n! \approx \sqrt{2\pi}\left(\frac{x+\frac{1}{2}-\frac{\frac{1}{24}}{x+\frac{1}{2}}+\frac{\frac{19}{5760}}{(x+\frac{1}{2})^3}}{e}\right)^{x+\frac{1}{2}}=\nu_n \tag{4.5}$$

where we take *i*=2, *i*=4 and *i*=8 in (4.1), respectively. Combining Theorem 1.2, we have **TABLE 1**.

TABLE 1. **Comparison for eight approximations.**

n	$(\alpha_n - n!)/n!$	$(\beta_{1,n} - n!)/n!$	$(\gamma_{13,n} - n!)/n!$	$(\nu_n - n!)/n!$	$(\theta_{2,n} - n!)/n!$	$(\theta_{4,n} - n!)/n!$	$(\theta_{8,n} - n!)/n!$
50	8.2540×10^{-4}	7.0647×10^{-8}	4.6003×10^{-13}	2.6852×10^{-12}	1.9085×10^{-12}	1.6402×10^{-19}	1.6518×10^{-31}
500	8.3254×10^{-5}	7.0902×10^{-11}	5.7555×10^{-18}	2.8089×10^{-17}	1.9966×10^{-17}	1.7808×10^{-28}	1.9432×10^{-48}
1000	4.1647×10^{-5}	8.8645×10^{-12}	1.8191×10^{-19}	8.7998×10^{-19}	6.2550×10^{-19}	3.4939×10^{-31}	1.4953×10^{-53}
1500	2.7769×10^{-5}	2.6267×10^{-12}	2.4045×10^{-20}	1.1598×10^{-19}	8.2439×10^{-20}	9.1020×10^{-33}	1.5220×10^{-56}
2000	2.0828×10^{-5}	1.1082×10^{-12}	5.7168×10^{-21}	2.7534×10^{-20}	1.9571×10^{-20}	6.8393×10^{-34}	1.1457×10^{-58}
3000	1.3887×10^{-5}	3.2836×10^{-13}	7.5424×10^{-22}	3.6273×10^{-21}	2.5784×10^{-21}	1.7804×10^{-35}	1.1645×10^{-61}

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