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MINIMUM ENERGY CONTROL OF Δ-DIFFERENTIABLE POSITIVE MATRIX SYLVESTER DYNAMICAL SYSTEMS

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ABSTRACT

The minimum energy control for the positive matrix sylvester dynamical system on time scales is formulated and obtain the solution. Also develop sufficient conditions for the existence of solution of the problem is proposed. Mathematics subject classification: 49K15, 93B05 and 37N35.

Key words: Time scales, Kronecker product, Minimum energy control, Fundamental matrix.

INTRODUCTION

The study of positive matrix dynamical systems on time scales is an interesting area of current research. Hilger in 1990 introduced time scales to unify and extend the theory of differential equations, difference equations and other differential systems defined over non empty closed subset of real line[1]. The two main objectives of this paper are (i) to develop the theory and methods to formulate the problem and solve dynamical system on time scales (ii) to develop the sufficient conditions for existence of solution to the problem.

Consider the time varying linear matrix Sylvester dynamic system

$$X^{\Delta}(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) + F_1(t)U(t)F_2^*(t) \qquad X(t_0) = X_0 \quad \ldots(1.1)$$

where $X(t)$ is an $n \times n$ matrix, $U(t)$ is $m \times n$ input piecewise rd-continuous matrix called control. Here $A(t)$, $B(t)$, and $F_1(t)$ are $n \times n$, $n \times n$, and $n \times m$ rd-continuous matrices respectively. $F_2(t)$ is a rd-continuous matrix of order $n \times n$ and $\mu(t)$ is a graininess function.

This paper is organized as follows. In section 2, we study some basic properties of Kronecker product of matrices and develop preliminary results by converting the given problem into a Kronecker product problem. The solution to the corresponding initial value

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problem obtained in terms of two transition matrices of the systems $X^{\Delta}(t) = A(t)X(t)$ and $X^{\Delta}(t) = B^{*}(t)X(t)$ by using the standard technique of variation of parameters[2]. Also the minimum energy control problem of the matrix positive time varying dynamical system is formulated and obtain its solution.

In Section 3, we address the sufficient conditions for the existence of solution of the problem are established and minimum value of the performance index are also presented.

Positive matrix sylvester dynamical system

In this section, we give a short over view on some basic results on the time scales and Kronecker product techniques that are important for the present treatment of minimum energy control of Kronecker product sylvester systems on time scales.

Definition 2.1[3] If P, $Q \in C^{n\times n}$ are two square matrices of order 'n' then their Kronecker product (or direct product or tensor product) is denoted by $P \otimes Q \in C^{n^2\times n^2}$ is defined to be partition matrix

$$P \otimes Q = \begin{bmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1}Q & p_{n2}Q & \cdots & p_{nn}Q \end{bmatrix}$$

We shall make use of vector valued function denoted by Vec P of a matrix P $= \{p_{ij}\} \in C^{n\times n}$ defined by –

$$\hat{P} = \text{Vec}P = \begin{bmatrix} P_{.1} \\ P_{.2} \\ \vdots \\ \vdots \\ P_{.n} \end{bmatrix}$$

where $P_{.j=}\begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ \vdots \\ p_{nj} \end{bmatrix} 1 \le j \le n$ it is clear that VecP is of order n^2.

The Kronecker product has the following properties[3].

1. $(P \otimes Q)^{*} = P^{*} \otimes Q^{*}$ (P^{*} denotes the transpose of P)

2. $(P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}$

3. The mixed product rule $((P \otimes Q)((M \otimes N) = (PM \otimes QN)$. This rule holds good, provided the dimension of the matrices are such that the various expressions exist.

4. If P(t) and Q(t) are matrices, then $(P \otimes Q)' = P' \otimes Q' (' = d/dt)$

5. Vec (PYQ) = $(Q^{*} \otimes P)$ Vec Y

6. If P and Q are matrices both of order n×n then

   (i) $\text{Vec}(PX) = (I_n \otimes P)\text{Vec}X$

   (ii) $\text{Vec}(XP) = (P^{*} \otimes I_n)\text{Vec}X$

A time scale T is an arbitrary non empty closed subset of the real numbers. The calculus on time scales was introduced by Aulbach and Hilger[1,4] in order to create a theory that can unify and extend discrete and continuous analysis.

For general introduction to the calculus of time scales we refer reader to the textbooks by Bohner and Peterson[5]. Here we gave only those notations and facts connected to time scales, which we need for our purpose in this paper.

A Timescale T is a closed subset of R; and examples of time scales include N; Z; R, Fuzzy sets etc. The set Q = $\{t \in R/Q, 0 \le t \le 1\}$ are not time scales. Time scales need not necessarily be connected. In order to overcome this deficiency, we introduce the notion of jump operators. Forward (backward) jump operator σ(t)of t for t < sup T (respectively ρ(t) at t for t >inf T) is given by $\sigma(t) = \inf\{s \in T : s > t\}$, $\rho(t) = \sup\{s \in T : s < t\}$, for all t ∈ T. The graininess function $\mu : T \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. Throughout we assume that T has a topology that it inherits from the standard topology on the real number R. The jump operators σ and ρ allow the classification of points in a time scale in the way: If σ(t) > t, then the point t is called right scattered ; while if ρ(t) < t, then t is termed left scattered. If t < sup T and σ(t) = t, then the point ‘ t’ is called right dense: while if t > inf T and ρ(t) = t, then we say ‘t’ is left-dense. We say that f : T → R is rd-continuous provided f is continuous at each right-dense point of T and has a finite left-sided limit at each left-dense point of T and will be denoted by Crd.

A function $f : T \to T$ is said to be differentiable at $t \in T^k = \{T \setminus (\rho\rho(t)\max(), \max t)\}$

if $\lim_{\sigma(t)\to s} \frac{f((\sigma((\sigma - f(s))}{\sigma(t) - s}$ where s∈T-{ σ(t)} exist and is said to be differentiable on T provided it is differentiable for each t∈T^k. A function F : T → T, with

F^Δ (t) = f(t) for all t∈T^k is said to be integrable, if $\int_s^t f(\tau(\tau) = F(t) - F(s)$

where F is anti derivative of f and for all s, t ∈T. Let f: T → T, and if T=R and a, b ∈T, then f^Δ (t)=f'(t) and $\int_a^b f(t)dt = \int_a^b f(t)\Delta t.$

If T=Z, then f^Δ (t) = Δf(t)=f(t+1)-f(t) and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k) & if \quad a < b \\ 0 & if \quad a = b \\ \sum_{k=b}^{a-1} f(k) & if \quad a > b \end{cases}$$

If f, g: T → X(X is a Banach space) be differentiable in t∈T^k. Then for any two scalars α, β the mapping α f+β g is differentiable in t and further we have:

(i) $(\alpha f+\beta g)^\Delta(t)= \alpha f^\Delta(t) +\beta g^\Delta(t)$

(ii) $(fg)^\Delta(t) =(f)^\Delta(t)g(t)+f(\sigma(t)) g^\Delta(t)$

(iii) $f(\sigma(t)) =f(t)+ \mu (t)f^\Delta(t)$

(iv) $(kf)^\Delta(t) = k f^\Delta(t)$, for any scalar k.

If f is Δ-differentiable, then f is continuous. Also if t is right scattered and f is continuous at t then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

Now by applying the Vec operator to the Δ-differentiable matrix dynamical system (1.1) and using Kronecker product properties[3], we have –

$$Z^{\Delta}(t) = G(t)Z(t) + [F_2 \otimes F_1]\hat{U}(t); \qquad Z(t_0) = Z_0 \qquad \ldots(2.1)$$

where Z(t) = Vec X(t), $\hat{U}(t) = \text{Vec } U(t)$, and G(t) $=[B^* \otimes I + I \otimes A + \mu(t)(B^* \otimes A)]$, is a $n^2 \times n^2$ matrix. Let A(t) and B(t) be regressive and rd-continuous. From the definition of Kronecker product $G : T^k \rightarrow R^{n^2}$ is regressive and rd-continuous.

When T = R, the equation (2.1) is equivalent to

$$Z'(t) = G(t)Z(t) + [F_2 \otimes F_1](t)\hat{U}(t); \qquad Z(t_0) = Z_0$$

and when T = Z, the equation (2. 1) is equivalent to

$$\Delta Z(n) = G(n)Z(n) + [F_2 \otimes F_1](n)\hat{U}(n); \qquad Z(n_0) = Z_0$$

System (2.1) is called the Kronecker product system associated with (1.1).

Remark 2.1[2] It is easily seen that, if X(t) is the solution of (1.1) then VecX(t)= Z(t) is the solution of (2.1) and vice-versa.

Now, we confine our attention to corresponding homogeneous matrix dynamical system (2.1) on time scales is –

$$Z^{\Delta}(t) = G(t)Z(t) \qquad \ldots(2.2)$$

Definition 2.1[2] Let A and B are rd-continuous matrices on time scale T, then

$$(A \otimes B)^{\Delta}(t) = A^{\Delta}(t) \otimes B(t) + A(\sigma(\sigma(t \otimes B^{\Delta}(t)$$

$A^{\Delta}(t)$ is the delta derivative of A, t is from a time scale T.

Lemma 2.1[7] Let $\varphi_1(t,s)$ and $\varphi_2(t,s)$ denote state transition matrices of the systems

$X^{\Delta}(t) = A(t)X(t)$ and $X^{\Delta}(t) = B^*(t)X(t)$ respectively. Then the matrix $\phi(t,s)$ defined by

$$\phi(t,s)=\phi_2(t,s)\otimes\phi_1(t,s) \qquad \ldots(2.3)$$

is the state transition matrix of (2.2) and every solution of (2.2) is of the form

$Z(t)=\phi(t,s)C$ (where C is any constant vector of order n^2).

Proof. Consider

$$\phi^{\Delta}(t,s)=\phi_2^{\Delta}(t,s)\otimes\phi_1(t,s)+\phi_2(\sigma(t),s)\otimes\phi^{\Delta}{}_1(t,s)$$

$$=B^*\phi_2(t,s)\otimes\phi_1(t,s)+(1+\mu(t)B^*)\,\phi_2(t,s)\otimes A\phi_1(t,s)$$

$$=[(B^*\otimes I_n)(\phi_2(t,s)\otimes\phi_1(t,s))+(\phi_2(t,s)+\mu(t)B^*\phi_2(t,s))\otimes(A\phi_1(t,s))$$

$$=(B^*\otimes I_n)(\phi_2(t,s)\otimes\phi_1(t,s))+(\phi_2(t,s)\otimes(A\phi_1(t,s))+\mu(t)B^*\phi_2(t,s))\otimes(A\phi_1(t,s))$$

$$=[(B^*\otimes I_n)(\varphi_2(t,s)\otimes\varphi_1(t,s))+(I_n\otimes A)(\varphi_2(t,s)\otimes\varphi_1(t,s))$$
$$+\mu(t)(B^*\otimes A)(\varphi_2(t,s)\otimes\varphi_1(t,s))$$

$$=[(B^*\otimes I_n)+(I_n\otimes A)+\mu(t)(B^*\otimes A)](\phi_2(t,s)\otimes\phi_1(t,s))$$

$$=G\phi(t,s)$$

Also $\phi(t,t)=\phi_2(t,t)\otimes\phi_1(t,t)=I_n\otimes I_n=I_{n^2}$

hence $\phi(t,s)$ is the transition matrix of (2.2). Moreover it can be easily seen that $\phi(t,s)$ is a solution of (2.2) and every solution of (2.2) is of this form.

Theorem 2.1[2] Let $\phi(t,s)=\phi_2(t,s)\otimes\phi_1(t,s)$ be a transition matrix of (2.2), then the unique solution of (2.1), subject to the initial condition $Z(t_0)=Z_0$ is –

$$Z(t)=\phi(t,t_0)[Z_0+\int_{t_0}^{t}\phi(t_0,\sigma(s))(F_2\otimes F_1)(s)\hat{U}(s)\Delta s] \qquad \ldots(2.4)$$

Lemma 2.2[8] The fundamental matrix satisfies $\varphi(t,t_0)\in T_+^{n^2\times n^2}$ for $t\geq t_0$ if and only if the off-diagonal entries $g_{i,j}$, $i\neq j$, $i,j=1,2,..n$ of the matrix $G(t)$ satisfy the condition

$$\int_{t_0}^{t} g_{i,j}(\tau)d\tau \geq 0 \text{ for } i \neq j, i, j = 1,2,...n$$

Definition 2.2[8] The system (2.1) is called the (internally) positive if and only if Z(t) $Z(t) \in T_+^{n^2}, t \geq t_0$. for any initial condition $Z(t_0) = Z_0 \in T_+^{n^2}, t \geq t_0$. and all inputs $\hat{U}(t) \in T_+^{mn}, t \geq t_0$.

Theorem2.2[9] The time-varying linear Kronecker product system (2.1) is positive if and only if the off-diagonal elements of the matrix G(t) satisfy the condition (2.5) and

$$(F_2 \otimes F_1) \in T_+^{n^2 \times nm}$$

Definition 2.3 [9] The system(2.1) is called reachable in time t_f to t_0 if for any given final state $Z_f \in T_+^{n^2}$, *for* $t \in [t_0, t_f]$ that steers the state of the system from zero initial state $Z(t_0)=Z_0$.

Definition 2.4[8] A real square matrix is called monomial if each of its row and each of its column contains only one positive entry and the remaining entries are zero.

Theorem 2.3. The positive system (2.1) is reachable in time t_f to t_o if and only if

$$R(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s)(F_2 \otimes F_1)^*(s)\boldsymbol{\phi}^*(t_f, \boldsymbol{\sigma}(s))\Delta s \qquad \ldots(2.6)$$

is a monomial matrix. The input vector which steers the state of the system (2.1) from $Z(t_0)=Z_0$ to the state Z_f is given by

$$\hat{U}(t) = -(F_2 \otimes F_1)^*(t)\boldsymbol{\phi}^*(t_f, \boldsymbol{\sigma}(s))R^{-1}(t_f, t_0)\{Z_0 - \boldsymbol{\phi}(t_0, t_f)Z_f\} \quad t \in [t_0, t_f] \quad \ldots(2.7)$$

Proof: If the matrix $R(t_0, t_f)$ is monomial if and only if $R^{-1}(t_0, t_f)$ is the inverse matrix

$R(t_0, t_f)$. Here the input $\hat{U}(t) \in T_+^{mn}$ defined by (2.7) steers the state of the system from $Z(t_0)=Z_0$ to the state Z_f . Substituting (2.7) into (2.4) for $t = t_f$ and $Z(t_0)=Z_0$ we get

$$Z(t_f) = \phi(t_f, t_0)[Z_0 - \int_{t_0}^{t_f} \phi(t_f, \sigma(s))(F_2 \otimes F_1)(s)(F_2 \otimes F_1)^*(s)\phi^*(t_f, \sigma(s)) R^{-1}(t_0, t_f)\{Z_0 - \phi(t_0, t_f)\}Z_f]\Delta s$$

$$= \phi(t_f, t_0)\, \phi(t_0, t_f) = Z_f .$$

Hence the positive system (2.1) is reachable in time t_f to t_0 if and only if the matrix (2.6) is monomial.

Minimum energy control problem

Consider the matrix Sylvester dynamical system (2.1) reachable in time t_f to t_0. If the system is reachable in time $t \in [t_0, t_f]$, then there exists many different inputs $\hat{U}(t) \in T_+^{mn}$ that steers the state of the system from $Z(t_0)=Z_0=0$ to $Z_f = Z(t_f) \in T_+^{n^2}$. Among these inputs we are looking for an input $\hat{U}(t) \in T_+^{mn}$ that minimizes the performance index

$$I(\hat{U}(t)) = \int_{t_0}^{t_f} \hat{U}^T(s)(I \otimes Q)\hat{U}(s)\Delta s \tag{3.1}$$

where $Q \in T_+^{m\times m}$ is a symmetric positive defined matrix and $Q^{-1} \in T_+^{m\times m}$.

The minimum energy control problem for the positive time varying linear systems (2.1) can be stated as follows: Given the matrices G(t), $[F_2 \otimes F_1]$ and Q of the performance index (3.1), $Z_f \in T_+^{n^2}$, t_0 and $t_f > 0$, find an input $\hat{U} \in T_+^{mn}$ for $t \in [t_0, t_f]$ that steers the state vector of the system from $Z_0 = 0$ to Z_f and minimizes the performance index (3.1).

Now we define the matrix for solving the problem

$$V = V(t_f, Q) = \int_{t_0}^{t_f} \phi(t_f, \sigma(s))(F_2 \otimes F_1)(s)(I \otimes Q)^{-1}(F_2 \otimes F_1)^*(s)\phi^*(t_f, \sigma(s))\Delta s \quad \ldots(3.2)$$

from (3.2) and Theorem 2.3 it follows that the matrix (3.2) is monomial if and only if the fractional positive dynamical system (2.1) is reachable in time $[t_0, t_f]$. In this case, we define the input –

$$\hat{U}_1(t) = -(I \otimes Q)^{-1}(F_2 \otimes F_1)^*(t)\boldsymbol{\phi}^*(t_f, \boldsymbol{\sigma}(s))V^{-1}(t_f, t_0)\{Z_0 - \boldsymbol{\phi}(t_0, t_f)Z_f\}\ \mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_\mathrm{f}] \quad \ldots(3.3)$$

The input (3.3) satisfies the condition $\hat{U}_1 \in T_+^{mn}$ for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_\mathrm{f}]$ if

$$Q^{-1} \in T_+^{m \times m}.\text{and } V^{-1} Z_f \in T_+^{n^2} \quad \ldots(3.4)$$

Theorem 3.1. Let the positive matrix Kronecker product dynamical system (2.1) be reachable in time [t_0, t_f] and let $\hat{U}_2 \in T_+^{mn}$ for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_\mathrm{f}]$ be an input that steers the state of the positive system(2.1) from Z_0 to Z_f and minimizes the performance index (3.1), i.e., $I(\hat{U}_1) \le I(\hat{U}_2)$.

The minimal value of the performance index (3.1) is equal to –

$$I(\hat{U}_1) = Z_f^T V^{-1} Z_f \quad \ldots(3.5)$$

Proof: If the conditions (3.4) holds then the input (3.3) is well defined and $\hat{U}_1 \in T_+^{mn}$ for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_\mathrm{f}]$. Now we prove the input steers the state of the system from

$Z_0 = 0$ to Z_f .

Substitute (3.3) into (2.4) for $t=t_f$ and $Z(t_0)=Z_0=0$, we get

$$Z(t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s)\hat{U}_1(s)\Delta s.$$

$$= \int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s) \times [-(I \otimes Q)^{-1}(F_2 \otimes F_1)^*(t)\boldsymbol{\phi}^*(t_f, \boldsymbol{\sigma}(s))V^{-1}(t_f, t_0)\{Z_0 - \boldsymbol{\phi}(t_0, t_f)Z_f\}]\Delta s.$$

$$= \boldsymbol{\phi}(t_f, t_0)\ \boldsymbol{\phi}(t_0, t_f) = Z_\mathrm{f}$$

Since (3.2) holds. Assume that the inputs $\hat{U}_1(t)$ and $\hat{U}_2(t)$, $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_\mathrm{f}]$ steers the state of the system from $Z(t_0) = Z_0 = 0$ to Z_f.

Hence

$$Z_f = \int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s)\hat{U}_2(s)\Delta s.$$
$$= \int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s)\hat{U}_1(s)\Delta s. \quad \ldots(3.7)$$

or

$$\int_{t_0}^{t_f} \boldsymbol{\phi}(t_f, \boldsymbol{\sigma}(s))(F_2 \otimes F_1)(s)[\hat{U}_2 - \hat{U}_1(s)]\Delta s = 0. \quad \ldots(3.8)$$

Taking transpose of (3.8) and post multiply with $V^{-1} Z_f$ we get

$$\int_{t_0}^{t_f} [\hat{U}_2 - \hat{U}_1(s)]^* (F_2 \otimes F_1)^* (s)\boldsymbol{\phi}^* (t_f, \boldsymbol{\sigma}(s))\Delta s V^{-1} Z_f = 0.$$

Substitution of (3.3) into (3.9) yields

$$\int_{t_0}^{t_f} [\hat{U}_2 - \hat{U}_1(s)]^* (F_2 \otimes F_1)^* (s)\boldsymbol{\phi}^* (t_f, \boldsymbol{\sigma}(s))\Delta s V^{-1} Z_f.$$
$$= \int_{t_0}^{t_f} [\hat{U}_2 - \hat{U}_1(s)]^* (I \otimes Q)\hat{U}_1 \Delta s = 0. \quad \ldots(3.9)$$

From (3.9), it is easy to verify that

$$\int_{t_0}^{t_f} [\hat{U}_2 {}^* (I \otimes Q)\hat{U}_2 \Delta s < \int_{t_0}^{t_f} [\hat{U}_1 {}^* (I \otimes Q)\hat{U}_1 \Delta s + .\int_{t_0}^{t_f} [\hat{U}_2 - \hat{U}_1]^* (I \otimes Q)[\hat{U}_2 - \hat{U}_1]\Delta s. \quad \ldots(3.10)$$

From (3.10) it follows that $I(\hat{U}_1) \leq I(\hat{U}_2)$. Since the second term in the right hand side of the inequality is nonnegative.

To find the minimal value of the performance index (3.1) we substitute (3.3) into (3.1) and we obtain –

$$I(\hat{U}_1(t)) = \int_{t_0}^{t_f} \hat{U}_1^T(s)(I \otimes Q)\hat{U}_1(s)\Delta s$$

$$= Z_f^T V^{-1} \int_{t_0}^{t_f} \phi(t_f, \sigma(s))(F_2 \otimes F_1)(s) \times [-(I \otimes Q)^{-1}(F_2 \otimes F_1)^*(t)\phi^*(t_f, \sigma(s))V^{-1}(t_f, t_0)\{Z_0 - \phi(t_0, t_f)Z_f\}]\,\Delta s$$

$$= Z_f^T V^{-1} Z_f.$$

since (3.2) holds.

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