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GENERALIZED OPERATIONAL RELATIONS AND PROPERTIES OF FRACTIONAL HANKEL TRANSFORM

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ABSTRACT

Namias[4] had defined fractionalization of conventional Hankel transform, using the method of eigen values and studied and open the door for the research in fractional integral transform. This paper studies the fractionalization of generalized Hankel transform, as given by Zemanian[5]. We referred it as fractional Hankel transform.

First we introduce fractional Hankel transform in the generalized sense. Generalized operational relations are derived that can be used to solve certain classes of ordinary and partial differential equations. Lastly the values of fractional Hankel transform are obtained for some special functions.

INTRODUCTION

Fourier analysis is one of the most frequently used tools in signal processing and is used in many other scientific disciplines. In the mathematics literature a generalization of the Fourier transform known as the fractional Fourier transform was proposed some years ago[1,3]. Although potentially useful for signal processing applications, the fractional Fourier transform has been independently reinvented by a number of researchers.

L. B. Almeida[1] had briefly introduced the fractional Fourier transform. He discussed the main properties and presented the new results including the fractional Fourier transform. Also represented simple relationship of the fractional Fourier transform with several time-frequency representations that supports the interpretation of it as a rotation operator.

Fiona H. Kerr[2] had defined the fractional Hankel transform with parameter α of *f(x)* denoted by H_α *f(x)* perform a linear operation given by the integral transform.

$$\left[H_\alpha f(x)\right](y) = \int_0^\infty f(x).K_\alpha(x,y)dx ,$$

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where

$$K_\alpha(x,y) = A_{\nu,\alpha}\exp\left(-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}} J_\nu\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)$$

$= \delta(x-y)$ for $\alpha = 0$ & $\alpha = 2\pi$, and

$$A_{\nu,\alpha} = \left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}\exp\left(i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(\nu+1)\right),\ \hat{\alpha} = \sin\alpha \qquad \ldots(1)$$

$f \in L^2(R^+)$, $\alpha \in R$ and $\nu > -1$

The above fractional Hankel transform is the generalization of the Hankel transform given as in[5],

$$H(f(x)) = \int_0^\infty \sqrt{xy}\, f(x) J_\nu(xy)\,dx$$

For the parameter $\alpha = \pi$, the fractional Hankel transform reduces to the above Hankel transform.

This paper is organized as follows. Section II presents the fractional Hankel transform with parameter α in the sense of generalized function and its interpretation as rotation operator. In section III, we give generalized operational relation. Section IV lists some properties of fractional Hankel transform. In section V we give fractional Hankel transform of some functions, lastly section VI concludes. Notaion and terminology is as used in Zemanian[5].

Franctional hankel transform in the generalized sense

First we define,

The testing function space E: An infinitely differentiable complex valued function ψ on R^n belongs to E (R^n) or E if for each compact set $K \subset S_a$ where $S_a = \{x \in R^n, |x| \le a, a > 0\}$, $k \in N^n$,

$$\gamma_{K,k}(\psi) = \sup_{x\in K}\left|D^k\psi(x)\right| < \infty$$

Clearly E is complete and so a Frechet space.

Moreover we say that f is a fractional transformable if it is a member of E' (the dual space of E).

The generalized fractional Hankel transform

It is easily seen that for each $x \in R^n$ and $0 < \alpha < 2\pi$, the function $K_\alpha(x,y)$ belongs to E as a function of x. Hence the fractional Hankel transform of $f \in E'$ can be defined by

$$[H_\alpha f(x)](y) = H_\alpha(y) = \langle f(x), K_\alpha(x,y)\rangle, \qquad \ldots(2)$$

where $K_\alpha(x,y)$ is as given by (1), then the right hand side of (2) has a meaning as a application of $f \in E'$ to $K_\alpha(x,y) \in E$.

Generalized operational relation of fractional hankel transform

As is well known an operational calculus can be based on the usual Hankel transform. We derive operational relations involving first derivatives and operational relations having second derivatives.

Operational relations involving first derivatives

Theorem: 3.1.1: If [illegible] denotes fractional Hankel transform of f(t) then -

$$H_\alpha\left(\frac{df}{dx}\right) = -\frac{1}{2}H_\alpha\left(\frac{f}{x}\right) + i\cot\frac{\alpha}{2}H_\alpha(fx) - \frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha(f)$$

Proof: We derive operational relation transform involving $H_\alpha\left(\frac{df}{dx}\right)$ obtain by inserting $\frac{df}{dx}$, instead of $f(x)$ in the integral representation (*2*), we then integrate by parts,

$$H_\alpha\left(\frac{df}{dx}\right) = A_{v,\alpha}\exp\left\{-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right\}\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\frac{df}{dx}dx$$

$$= -A_{v,\alpha}\exp\left\{-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right\}\left\{\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)x^{\frac{1}{2}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)f\,dx\right.$$

$$+\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(-ix\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)f\,dx$$

$$\left.+\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\frac{f}{x}dx\right\}$$

$$= -H_\alpha\left(\frac{f}{2x}\right) + i\cot\frac{\alpha}{2}H_\alpha(f\,x) + vH_\alpha\left(\frac{f}{x}\right) - A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)$$

$$\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)J_{v-1}\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\frac{f}{x}dx$$

after some straight forward steps we obtain,

$$H_\alpha\left(\frac{df}{dx}\right)=-\frac{1}{2}H_\alpha\left(\frac{f}{x}\right)+i\cot\frac{\alpha}{2}H_\alpha(fx)-\frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha(f) \qquad \ldots(4)$$

is the operational relation involving first derivative, in (4) replace *f* by *fx*, $\frac{df}{dx}\to f+\frac{df}{dx}$ gives,

$$H_\alpha\left(x\frac{df}{dx}\right)=-\frac{3}{2}H_\alpha(f)+i\cot\frac{\alpha}{2}H_\alpha\left(fx^2\right)-\frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha(fx) \qquad \ldots(5)$$

replacing *f* by $\frac{f}{x}$ in (3), $\frac{df}{dx}\to\frac{1}{x}\frac{df}{dx}-\frac{f}{x^2}$ gives

$$H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right)=\frac{1}{2}H_\alpha\left(\frac{f}{x^2}\right)+i\cot\frac{\alpha}{2}H_\alpha(f)-\frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha\left(\frac{f}{x}\right) \qquad \ldots(6)$$

Operational relations involving second derivatives :

We now calculate second derivative $H_\alpha\left(\frac{d^2f}{dx^2}\right)$ by inserting $\frac{df}{dx}$ in place of *f* in the equation (4), we obtain,

$$H_\alpha\left(\frac{d^2f}{dx^2}\right)=-\frac{1}{2}H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right)+i\cot\frac{\alpha}{2}H_\alpha\left(x\frac{df}{dx}\right)-\frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha\left(\frac{df}{dx}\right)$$

$$=-\frac{1}{4}H_\alpha\left(\frac{f}{x^2}\right)+\frac{y}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha\left(\frac{f}{x}\right)-\frac{2yi\cot\frac{\alpha}{2}}{\left|\sin\frac{\alpha}{2}\right|}H_\alpha(fx)-\cot^2\frac{\alpha}{2}H_\alpha\left(fx^2\right)$$

$$+\left(\frac{y^2}{\sin^2\frac{\alpha}{2}}-\frac{i}{2}\cot\frac{\alpha}{2}\right)H_\alpha(f)\ .$$

Properties of fractional hankel transform

We prove the following properties of fractional Hankel transform,

$$H_\alpha[f(cx)]=\frac{A_{v,\alpha}}{cA_{v,\beta}}\exp\left[-\frac{i}{2}\left(\frac{y^2}{c^2}\frac{\left|\sin\frac{\beta}{2}\right|^2}{\left|\sin\frac{\alpha}{2}\right|^2}\right)\left(c^4\frac{\left|\sin\frac{\alpha}{2}\right|^2}{\left|\sin\frac{\beta}{2}\right|^2}-\frac{1}{c^2}\right)\cot\frac{\beta}{2}\right]H_\beta(f(x))\left(\frac{y}{c}\frac{\left|\sin\frac{\beta}{2}\right|}{\left|\sin\frac{\alpha}{2}\right|}\right)$$

where $\cot\frac{\beta}{2}=\frac{1}{c^2}\cot\frac{\alpha}{2}$

Proof :

$$H_\alpha[f(cx)] = \int_0^\infty f(cx) A_{v,\alpha} \exp\left(-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right) dx$$

$$= \int_0^\infty f(t) A_{v,\alpha} \exp\left(-\frac{i}{2}\left(\frac{x^2}{c^2}+y^2\right)\cot\frac{\alpha}{2}\right)\left(\frac{ty}{c\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{ty}{c\left|\sin\frac{\alpha}{2}\right|}\right)\frac{dt}{c}, \text{ where } t = cx$$

$$= \frac{A_{v,\alpha}}{cA_{v,\beta}} \int_0^\infty f(t) A_{v,\beta} \exp\left(-\frac{i}{2}(t^2+c^2y^2+z^2-z^2)\cot\frac{\beta}{2}\right)\left(tz\frac{1}{\left|\sin\frac{\beta}{2}\right|}\right)^{\frac{1}{2}} J_v\left(tz\frac{1}{C\left|\sin\frac{\beta}{2}\right|}\right) dt ,$$

Where

$$z = \frac{y\left|\sin\frac{\beta}{2}\right|}{C\left|\sin\frac{\alpha}{2}\right|}$$

$$= \frac{A_{v,\alpha}}{cA_{v,\beta}} \exp\left(-\frac{i}{2}z^2\left(\frac{c^4\left|\sin\frac{\alpha}{2}\right|^2}{\left|\sin\frac{\beta}{2}\right|^2}-\frac{1}{c^2}\right)\cot\frac{\beta}{2}\right) H_\beta[f(x)]\left(\frac{y\left|\sin\frac{\beta}{2}\right|}{c\left|\sin\frac{\alpha}{2}\right|}\right),$$

$$= \frac{A_{v,\alpha}}{cA_{v,\beta}} \exp\left(-\frac{i}{2}\left(\frac{y^2\left|\sin\frac{\beta}{2}\right|^2}{c^2\left|\sin\frac{\alpha}{2}\right|^2}\right)\left(\frac{c^4\left|\sin\frac{\alpha}{2}\right|^2}{\left|\sin\frac{\beta}{2}\right|^2}-\frac{1}{c^2}\right)\cot\frac{\beta}{2}\right) H_\beta[f(x)]\left(\frac{y\left|\sin\frac{\beta}{2}\right|}{c\left|\sin\frac{\alpha}{2}\right|}\right)$$

$$H_\alpha\left(\exp\left(\frac{i}{2}x^2\right)f(x)\right) = A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\alpha\right)H_\alpha(f(x))\left(\frac{y}{|\sin\alpha|}\right)$$

the proof is trivial, hence omitted.

Transform of some common functions

The Hankel transform of some common functions are proved.

Result 1

$$\delta(x-\tau) = \left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}} \exp\left\{i\left(\frac{\alpha}{2}-\frac{\pi}{2}\hat{\alpha}\right)(v+1)-\frac{i}{2}(\tau^2+y^2)\cot\frac{\alpha}{2}\right\} J_v\left(\frac{\tau y}{\left|\sin\frac{\alpha}{2}\right|}\right)\left(\frac{\tau y}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}$$

the proof is trivial & hence omitted.

Result 2:

$$H_\alpha\left[x^{v+\frac{1}{2}}\exp\left(-ax^2\right)\right]=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\left(\frac{\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^{v+\frac{1}{2}}}{(2a)^{v+1}}\exp\left(\frac{-\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^2}{8a}\right)\right)$$

Re a > 0, Re v > -1.

Proof:

$$H_\alpha\left[x^{v+\frac{1}{2}}\exp\left(-ax^2\right)\right]=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\int_0^\infty\exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(x^{v+\frac{1}{2}}\exp\left(-ax^2\right)\right)$$

$(xy_1)^{\frac{1}{2}}J_v(xy_1)dx$, where $y_1=\frac{y}{\left|\sin\frac{\alpha}{2}\right|}$,

$$=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\int_0^\infty x^{v+\frac{1}{2}}\exp\left(-Bx^2\right)(xy_1)^{\frac{1}{2}}J_v(xy_1)dx$$

where $B=a+\frac{i}{2}\cot\frac{\alpha}{2}$

$$=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\left(\frac{\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^{v+\frac{1}{2}}}{(2a)^{v+1}}\exp\left(\frac{-\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^2}{8a}\right)\right)$$

Result 3:

$$H_\alpha\left[J_v(ax)x^{\frac{1}{2}}\right]=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\int_0^\infty\exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(x^{\frac{1}{2}}J_v(ax)\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)dx$$

Proof:

$$H_\alpha\left[J_v(ax)x^{\frac{1}{2}}\right]=A_{v,\alpha}\exp\left(-\frac{i}{2}y^2\cot\frac{\alpha}{2}\right)\int_0^\infty \exp\left(-\frac{i}{2}x^2\cot\frac{\alpha}{2}\right)\left(x^{\frac{1}{2}}J_v(ax)\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{\frac{1}{2}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)dx$$

$$=\frac{y}{\left|\sin\frac{\alpha}{2}\right|}A_{v,\alpha}\left[\frac{1}{y^{\frac{1}{2}}}\int_0^\infty x^{\frac{1}{2}}(xy)^{\frac{1}{2}}\cos\left(\frac{1}{2}\cot\frac{\alpha}{2}x^2\right)J_v(ax)J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)dx\right.$$

$$\left.-i\frac{1}{y^{\frac{1}{2}}}\int_0^\infty x^{\frac{1}{2}}(xy)^{\frac{1}{2}}\sin\left(\frac{1}{2}\cot\frac{\alpha}{2}x^2\right)J_v(ax)J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)dx\right]$$

$$=\frac{A_{v,\alpha}}{\left|\sin\frac{\alpha}{2}\right|}\frac{y^{\frac{1}{2}}}{\cot\frac{\alpha}{2}}J_v\left(\frac{ay}{\cot\frac{\alpha}{2}}\right)\exp\left(-i\left(\frac{a^2-\left(\frac{y}{\left|\sin\frac{\alpha}{2}\right|}\right)^2}{2\cot\frac{\alpha}{2}}\right)-\frac{v\pi}{2}\right)$$

CONCLUSION

We have introduced an extension of Hankel transform that is designated fractional Hankel transform. This linear transform depends on a parameter α and can be interpreted as a rotation by an angle α in scale-log phase modulation plane. When $\alpha=\pi$, the fractional Hankel transform coincides with the conventional generalized Hankel transform. We derive operational relation for first and second order derivative for fractional Hankel transform. Some properties of the fractional Hankel transform are given which coincides with corresponding properties for Hankel transform in special case. Fractional Hankel transform of some simple functions are also obtained.

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