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GENERALIZED LINEAR CANONICAL TRANSFORM OF COMPACT SUPPORT

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ABSTRACT

Linear Canonical Transform which has many applications in signal processing is the generalization of Fractional Fourier Transform with three more parameters. In this paper we define it on the spaces of generalized functions and discuss its analyticity and inverse. We have also proved some operation transform formulae for this generalization.

Key words: Linear canonical transform, Compact support.

INTRODUCTION

The fractionalization of a linear operator gave birth to many new ideas in the theory of fractional calculus. Namias[3] developed the method of Eigen function for defining fractional Fourier transform which is the generalization of Fourier Transform. Further generalization of Fractional Fourier Transform is Linear Canonical Transform. Comparing to the fractional Fourier transform with one extra degree of freedom and the Fourier transform without a parameter, the Linear canonical transform (LCT) is more flexible for its extra three degrees of freedom, without increasing the complexity of the computation. Pei and Ding[4] studied Eigen function of Linear canonical transform where as Alieva[1] described time frequency rotational property of Linear canonical transform.

Definition: The Linear Canonical Transform is defined as –

$$\text{LCT [f(t)]} = \int_{-\infty}^{\infty} f(t)\, K_A\,(u,t)\, dt \qquad \ldots(1.1)$$

where the kernel is –

$$\text{KA (u,t)} = \sqrt{-i}.e^{i\pi\frac{d}{b}u^2}, e^{i\pi\frac{1}{b}ut}, e^{i\pi\frac{a}{b}t^2}, b \neq 0$$

$$= \sqrt{-i}.e^{i\pi c d u^2}, f(du), b \neq 0 \qquad \ldots(1.2)$$

and a, b, c, d are real numbers with ad – bc = 1. The kernel can also be viewed as a 2 × 2 matrix with

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determinant 1 that is an element of the special linear space $SL_2(R)$, we use A = $\begin{bmatrix} a & a \\ c & c \end{bmatrix}$ with determinant ad-bc = 1.

In this paper we have studied Linear canonical transform in the analytical sense. In section 2 we define the transform on the spaces of generalized functions and discuss some properties of kernel. Section 3 is devoted to obtain inverse of this transform. Some operation transform formulae are developed in section 4 and the paper is concluded lastly in section 5.

Definition

The test function space E: An infinitely differentiable complex valued function ϕ on R^n belongs to $E(R^n)$ if for each compact set $K \subset S_\alpha$ where $S_\alpha = \{x: \in R^n, |x| \le a, a > 0\}$, $K \in R^n$,

$$\gamma_{E,k}(\phi) = \sup_{x \in K} \left| D^k \phi(x) \right| < \infty$$

In what follows $E(R^n)$ will denote the space of all $\phi \subset E(R^n)$ with support contained in S_α.

Note that the space E is complete and therefore a Frechet space. Let E' denotes the dual of E.

The Linear canonical transform on E': It is easily seen that for each $\zeta \in R^n$ and $0 \le \phi \le \frac{\pi}{2}$, the function $K_A(t, \zeta)$ belongs to E as a function of t.

Hence the Linear canonical transform of $f \in E'$ can be defined by -

$$[LCT[f(t)]](\zeta) = f_A(\zeta) = \langle f(t), K_A(t,\zeta) \rangle ,$$

where $K_A(t,\zeta)$ is given as in (1.2).

Proposition: Let $f \in E'(R^n)$ and let its Linear canonical transform be (1.1). Then $\left[D_\tau^k f(t)\right]$ is analytic on C^n if the supp $f \subset S_\pi = \{x : x \in R^n |x| \le a, a > 0\}$ Moreover then the linear canonical transform $f_A(\zeta)$ is differentiable and

$$D_\tau^k f_A(\zeta) = \left\langle f(t), D_\tau^k K_A(t,\zeta) \right\rangle$$

Proof: Let $\zeta : (\zeta_1, ..., \zeta_i, ..., \zeta_n) \in C^n$

We first prove that –

$$\frac{\partial}{\partial \zeta}[f_A(\zeta)] = \left\langle f(\zeta), \frac{\partial}{\partial \zeta} K_A(t,\zeta) \right\rangle$$

For fix $\zeta_i \ne 0$, choose two concentric circles C and C^1 with center at ζ_i and radii r and r_1 respectively, such that $0 < r < r_1 |\zeta_i|$. Let $\Delta\zeta_i$ be a complex increment satisfying $0 < |\Delta\zeta_i| < r$. 0 Consider,

$$\frac{[f_A(\zeta_i + \Delta\zeta_i)] - [f_A(\Delta\zeta_i)}{\Delta\zeta_i} - \left\langle f(t), \frac{\partial}{\partial \zeta_i} K_A(t,\zeta) \right\rangle = \left\langle f(t), \psi_{\Delta\zeta_i}(t,\zeta) \right\rangle$$

where $\psi_{\Delta\zeta_i} = \frac{1}{\Delta\zeta_i}[K_A(t,\zeta_1,....,\zeta_i+\Delta\zeta_i,....,\zeta_n) - K_A(t,\zeta)] - \frac{\partial}{\Delta\zeta_i}K_A(t,\zeta)$

We have by Cauchy Integral formula,

$$D_t^k K_{\Delta\zeta_i}(t) = \frac{1}{2\pi i} D_t^k \int K_A(t,\zeta)\left(\frac{1}{\Delta\zeta_i}\left(\frac{1}{z-\zeta_i-\Delta\zeta_i} - \frac{1}{z-\zeta_i}\right) - \frac{1}{(z-\zeta_i)^2}\right)dz$$

$$= \frac{\Delta\zeta_i}{2\pi i}\int \frac{D_t^k K_A(t,\overline{\zeta})}{(z-\zeta_i-\Delta\zeta_i)(z-\zeta_i)^2}\,dz$$

where $\zeta = (\zeta_1,...,\zeta_{i-1},\zeta_{i+1},....,\zeta_n)$

$$= \frac{\Delta\zeta_i}{2\pi i}\int \frac{M(t,\overline{\zeta})}{(z-\zeta_i-\Delta\zeta_i)(z-\zeta_i)^2}\,dz$$

But for all z$\in C_1$ and t restricted to a compact subset of R^n, $M(t,\overline{\zeta}) = D_t^k K_A(t,\overline{\zeta})$ is bounded by a constant K.

Therefore we have-

$$\left|D_t^k \psi_{\Delta\zeta_i}(t),\overline{\zeta})\right| \le \left|\Delta\zeta_i\right| K/(r_1-r)r_1$$

Thus, as $\left|\Delta\zeta_i\right| \to 0, D_t^k \Delta\zeta_i(t)$ tends to zero uniformly on the compact subsets of R^n, therefore it fellows that $\psi\Delta_{\zeta\ i}$ (t) converges in E (R^n) to zero. Since F $\in$ E we conclude that ( ) also tends to zero. Therefore, LCT$[f_A(\zeta)]$ is differentiable with respect to ζ_i. But this is true for all i = 1,2,….n hence LCT$[f_A(\zeta)]$ is analytic on C^n, and

$$D_\zeta^k\ LCT\ [f_A(\zeta)] = \left\langle f(t), D_\zeta^k K_A(t,\zeta)\right\rangle$$

Properties of the Kernel

$$K_{(a,b,c,d)}\ (t,u) = K_{(d,b,c,a)}\ (t,u)$$

$$K_{(a,b,c,d)}\ (\text{-}t,u) = K_{(a,b,c,d)}\ (t,-u)$$

Proof is very simple and hence omitted.

Inversion Formula: If LCT $[f(t)](u) = \left\langle f(t), K_A(u,t)\right\rangle$ where $K_A(u,t)$ is given by (1.2) and $x \in S_a = \{x : x \in R^n, \left| x \right| < a, a > 0\}$ then inverse of (1.1) is given by –

$$f(t) = \frac{1}{b}\left\langle [f_A(u)](t), \overline{K}_A(u,t)\right\rangle$$

where

$$\overline{K}_A(u,t) = \sqrt{-i}e^{i\pi\frac{a}{b}t^2}.e^{i2\pi ut}.e^{-i\pi\frac{d}{b}u^2}$$

It is possible to recover the function f by means of inversion formula.

$$LCT[f(t)](u) = \sqrt{-i}e^{i\pi\frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{i2\pi\frac{1}{b}ut} . e^{i\pi\frac{a}{b}t^2} f(t)\,dt$$

$$e^{i\pi\frac{d}{b}u^2} LCT[f(t)](u) = \sqrt{-i} \int_{-\infty}^{\infty} e^{i2\pi\frac{1}{b}ut} . e^{i\pi\frac{a}{b}t^2} f(t)\,dt$$

$$= \sqrt{-i} \int_{-\infty}^{\infty} e^{i2\pi\frac{1}{b}ut} . g(t)\,dt$$

Where $g(t) = e^{i\pi\frac{a}{b}t^2} f(t)\,dt$

$$= FT\,[g(t)]\left(\frac{2\pi u}{b}\right)$$

is the Fourier Transform of g(t) with argument $\left(\frac{2\pi u}{b}\right) = \eta$ (say)

$$\therefore e^{-i\pi\frac{d}{b}u^2}\ LCT\ [f(t)] \left(\frac{\eta b}{2\pi}\right) = \sqrt{-i}FT\,[g(t)](\eta)$$

Invoking the Fourier Inverse, we can write

$$G(t) = \sqrt{-i}\,\frac{1}{2\pi}\,G(\eta)e^{it\eta}d\eta$$

where $G(\eta) = e^{-i\pi\frac{d}{b}\left(\frac{\eta b}{2\pi}\right)^2} LCT[f(t)]\left(\frac{\eta b}{2\pi}\right)$ (say).

Therefore,

$$g(t) = \sqrt{-i}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\pi\frac{d}{b}\left(\frac{\eta b}{2\pi}\right)^2} LCT[f(t)]\left(\frac{\eta b}{2\pi}\right) . e^{-it\left(\frac{2\pi u}{b}\right)} . \frac{2\pi}{b}\,du$$

$$e^{-i\pi\frac{a}{b}t^2} f(t) = \sqrt{-i}\int_{-\infty}^{\infty} e^{-i\pi\frac{d}{b}u^2} . e^{i2\pi ut} . LCT\,[f(t)]\,(u)\,du$$

$$f(t) = \frac{1}{b}\sqrt{-i}\,e^{-i\pi\frac{a}{b}t^2}\int_{-\infty}^{\infty} e^{-i2\pi ut} . e^{-i\pi\frac{d}{b}u^2} . LCT[f(t)](u)\,du$$

$$f(t) = \frac{1}{b}\int_{-\infty}^{\infty} LCT\,[f(t)](u) . \overline{K}_A(u,t)\,du$$

Operation transform formulae for linear canonical transform

Proposition: If $\tau \in R$ then $LCT\ [f\ (t+\tau)](u) = LCT\,[f(t)](u + a\tau) . e^{i\pi\frac{a}{b}\tau^2(1-ad)}\ e^{i2\pi\frac{1}{b}u\tau(1-ad)}$

Proof: Now,

$$LCT\ [F(t+\tau)](u)=\sqrt{-i}e^{-i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-i2\pi\frac{1}{b}ut}.e^{-i2\pi\frac{a}{b}t^2}f(t+\tau)dt$$

$$=\sqrt{-i}.e^{-i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-i2\pi\frac{1}{b}u(T-\tau)}.e^{-i\pi\frac{a}{b}(T-\tau)^2}f(t)\,dT$$

$$=\sqrt{-i}.e^{-i\pi\frac{d}{b}u^2}.e^{-i2\pi\frac{1}{b}u\tau}.e^{-i\pi\frac{a}{b}\tau^2}\int_{-\infty}^{\infty}e^{-i2\pi\frac{1}{b}u\tau}.e^{-i2\pi\frac{a}{b}t\tau}.e^{-i\pi\frac{a}{b}t^2}f(t)\,dt$$

$$=\sqrt{-i}.e^{-i\pi\frac{d}{b}u^2}.e^{-i2\pi\frac{ad}{b}u\tau}.e^{-i\pi\frac{a^2d}{b}\tau^2}\int_{-\infty}^{\infty}e^{-i2\pi\frac{1}{b}(u+a\tau)t}.e^{-i2\pi\frac{a}{b}t^2}.e^{-i2\pi\frac{ad}{b}u\tau}.e^{-i\pi\frac{a^2d}{b}u\tau}.e^{-i\pi\frac{a}{b}\tau^2}.e^{-i2\pi\frac{1}{b}u\tau}f(t)\,dt$$

$$LCT[y(t)](u)=LCT[f(t)](u+a\tau).e^{i\pi\frac{a}{b}\tau^2(1-ad)}.e^{i2\pi\frac{1}{b}\tau^2(1-ad)}$$

Proposition: If $\nu\in R$ then LCT $LCT\,[y(t).e^{ivt}]u(t)=LCT[f(t)]\left(u-\frac{vb}{2\pi}\right)e^{\frac{-ivdb^2}{4\pi}+iduv}$

Proof: Let $y(t)=f(t)\,.\,e^{ivt}$ then its LCT is –

$$LCT[y(t)](u)=\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{i\pi\frac{a}{b}t^2}.e^{i2\pi\frac{1}{b}ut}.e^{ivt}f(t)\,dt$$

$$=\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{i2\pi\frac{a}{b}t^2}.e^{ivt}.e^{i\pi\frac{a}{b}t^2}.f(t)\,dt$$

$$=\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{i2\pi\frac{1}{b}\left(u-\frac{vb}{2\pi}\right)t}.e^{i\pi\frac{a}{b}t^2}.f(t)\,dt$$

$$=\sqrt{-i}e^{i\pi\frac{d}{b}\left(u-\frac{vb}{2\pi}\right)^2}\int_{-\infty}^{\infty}e^{i2\pi\frac{1}{b}\left(u-\frac{vb}{2\pi}\right)t}.e^{i\pi\frac{a}{b}t^2}.e^{iduv}.e^{\frac{-iv^2db}{4\pi}}f(t)\,dt$$

$$LCT\,[y(t)]\,(u)=LCT[f(t)]\left(u-\frac{vb}{2\pi}\right).e^{\frac{-ivdb^2}{4\pi}+iduv}$$

Proposition: $[f'](u)=-2\pi i\frac{a}{b}LCT[t.f(t)](u)+2\pi iu\frac{1}{b}LCT[f(t)](u)$

Let us make $s(t)=f'\,(t)$ then its LCT is –

$$LCT[s(t)](u)=\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{i2\pi\frac{1}{b}ut}.e^{i\pi\frac{a}{b}t^2}.f'(t)\,dt$$

$$=\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\left\{\left[e^{i2\pi\frac{1}{b}ut}.e^{i\pi\frac{a}{b}t^2}.f'(t)\,dt\right]_{-\infty}^{\infty}\right\}-\int_{-\infty}^{\infty}\frac{d}{dt}\left(e^{i2\pi\frac{1}{b}ut}.e^{i\pi\frac{a}{b}t^2}\right)f'(t)\,dt$$

$$= \left(2i\boldsymbol{\pi}\frac{a}{b}\right)\sqrt{-i}e^{i\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty} e^{i2\boldsymbol{\pi}\frac{1}{b}ut}.e^{i\pi\frac{a}{b}t^2}t.f(t)\,dt + \left(i2\boldsymbol{\pi}\frac{1}{b}u\right)\sqrt{-i}\int_{-\infty}^{\infty} e^{i2\boldsymbol{\pi}\frac{1}{b}ut}.e^{i\pi\frac{a}{b}t^2}f(t)\,dt$$

$$= \left(2i\boldsymbol{\pi}\frac{a}{b}\right)LCT[t.f(t)](u) + \left(i2\boldsymbol{\pi}\frac{1}{b}u\right)LCT[f(t)](u)$$

Here we note that the properties given by Almeida [2] for Fractional Fourier Transform are the special cases of these properties.

CONCLUSION

In this work we have just studied the introductory analytic part of generalized Linear Canonical Transform. We further wish to develop its properties as an operator so that it can be considered as an important tool to solve partial differential equation. Definitely this will be more useful and having wide applicability because it has three more parameters than that of Fractional Fourier Transform.

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REFERENCES

1. T. Alieva, M. J. Bastiaans and L. Stankovic, on Rotated Time-Frequency Kernels, IEEE Sig. Proc. Letters, 9, 387-381 (2002).
2. L. B. Almeida, An Introduction to Angular Fourier Transform, IEEE, 257- 260 (1993).
3. V. Namias, The Fractional Order Fourier Transform and its Applications in Quantum Mechanics, J. Inst. Math. Appl., 25, 241-265 (1980).
4. S. C. Pei and J. J. Ding, Eigen Function of Linear Canonical Transform, IEEE Trans. Sig. Proc., 50(1), 11-26 (2002).