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ANGULAR COMPLEX MELLIN TRANSFORM

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ABSTRACT

The angular (fractional) complex Mellin transform which is a generalization of the complex mellin transform has many applications in several areas including signal processing and optics. In this paper we have given inversion theorem for the generalized Fractional Complex Mellin transform. For that we first prove two lemmas. Lastly we have discussed some applications of this transform.

Key words: Complex mellin transform, Angular mellin transform.

INTRODUCTION

The fractional Fourier transform is intimately related to several indispensable concepts appearing in diverse areas. It plays an important role in the study of optical system known as Fourier optics. The brief account of its application is discussed in[4].

The fractional Fourier transform R^α is an extension of the ordinary Fourier transform and depends on the parameter α in the position-frequency plane. The one dimensional fractional Fourier transform with parameter α of *f(x)* denoted by $R^\alpha f(x)$[1] performs a linear operation, given by the integral transform,

$$\left[R^\alpha f(x)\right](\xi) = F_\alpha(\xi) = \int_{-\infty}^{\infty} K_\alpha(x,\xi)\, f(x)dx \qquad \ldots(1)$$

Where $K_\alpha(x,\xi)$ is the kernel as given in[1].

Bhosale[3] extended fractional Fourier transform to the distributions of compact support.

The fractional complex Mellin transform introduced in[5] is the generalization of the complex Mellin transform is as follows.

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In[2] the angular (fractional) Mellin transform which is a generalization of the complex Mellin transform had extended to the distribution of compact support using kernel method.

The testing function space *E* defined in[5] is as follows -

The testing function space E

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ or E if for each compact set $K \subset S_a$

where $S_a = \{x \in R^n, |x| \leq a, a > 0\}$, $k \in N^n$, $\gamma_{E,k}(\psi) = \sup_{x \in K} |D^k \psi(x)| < \infty$

Clearly *E* is complete and so a Frechet space.

Moreover we say that f is a fractional Mellin transformable if it is a member of *E'* (The dual space of *E*).

The fractional Mellin transform on E'

It is easily seen that for each $s \in R^n$ and $0 \leq \phi \leq \frac{\pi}{2}$, the function $K_\phi(x,s)$ belongs to *E* as a function of *x*.

Hence the fractional Mellin transform of $f \in E'$ can be defined by

$$\left[M^\phi f(x)\right](s) = M_\phi(s) = \left\langle f(x), K_\phi(x,s) \right\rangle \text{ ,} \qquad \ldots(1.1)$$

$$\text{where } K_\phi(x,s) = \sqrt{1 - i\cot\phi}\; e^{i\pi s^2 \cot\phi} . e^{i\pi(\ln x)^2 \cot\phi - i2\pi(\ln x)s.\csc\phi} \frac{1}{\sqrt{x}}, \qquad \ldots(1.2)$$

then the right hand side of (1.2) has a meaning as the application of $f \in E'$ to $K_\phi(x,s) \in E$.

The paper is organized as follows. Section II gives the inversion theorem with two lemmas. Some applications are given in Section III and Section IV concludes the paper

Notations and terminology used as in Zemanian[6].

Section II: Inversion theorem

Let $f \in \mathscr{E}'(R)$, $0 < \phi < \pi$, and supp $f \subset S_a$ where $S_a = \{x : x \in R, |x| \leq a,\ a > 0\}$ and let $M_\phi(\varsigma)$ be the generalized fractional complex Mellin transformation of *f* defined by,

$\left[M^\phi f(x)\right](\varsigma) = M_\phi(\varsigma) = \left\langle f(x), K_\phi(x,\varsigma) \right\rangle$, $K_\phi(x,\varsigma)$ as per (3.2.3).

Then for each $\psi \in \mathscr{E}$ we have,

$$\left\langle f(x), \psi(x) \right\rangle = \left\langle \frac{1}{\pi} \int_{-\infty}^{\infty} \overline{K_\phi(x,\varsigma)}\, M_\phi(\varsigma) d\varsigma, \psi(x) \right\rangle,$$

$$\text{where } \overline{K_\phi(x,\varsigma)} = \frac{\pi}{\sin\phi}(i\sin\phi)^{1/2} \exp\left(\frac{-i\phi}{2}\right) \exp\left\{-\frac{i\pi}{\sin\phi}\left[\left((\ln x)^2 + \varsigma^2\right)\cos\phi - 2\ln x.\varsigma\ \right]\right\} \frac{1}{\sqrt{x}}$$

Proof : To prove the inversion theorem, we have established the following lemmas to be used in the sequel.

Lemma 1: Let $\left[M^{\phi} f(x)\right](\varsigma) = M_{\phi}(\varsigma)$ for $0 < \phi < \pi$ and supp $f \subset S_a$

where $S_a = \{x : x \in R, |x| \leq a, \ a > 0\}$ for $\theta(x) \in Ɇ$,

$$\psi(\varsigma) = \int_{-\infty}^{\infty} \overline{K_{\phi}(x,\varsigma)}\, \theta(x) dx$$

Then for any fixed number r, $-\infty < r < \infty$

$$\int_{-r}^{r} \psi(\varsigma) \left\langle f(u), K_{\phi}(u,\varsigma) \right\rangle d\tau = \left\langle f(u), \int_{-r}^{r} \psi(\varsigma) K_{\phi}(u,\varsigma) d\tau \right\rangle \quad \ldots(2.1)$$

where $\varsigma = \sigma + i\tau \in C^n$ and u is restricted to a compact subset of R.

Proof : The case $\theta(x) = 0$ is trivial, hence consider $\theta(x) \neq 0$. It can be easily seen that,

$$\int_{-r}^{r} \psi(\varsigma)\, K_{\phi}(u,\varsigma) d\tau \, , \ \varsigma = \sigma + \mathrm{i}\tau$$

is a C^{∞} - function of u and it belongs to E. Hence the right hand side of (2.1) is meaningful.

To prove the equality, we construct the Riemann-sum for this integral and write,

$$\int_{-r}^{r} \left\langle f(u), K_{\phi}(u,\varsigma) \right\rangle \psi(\varsigma) d\tau$$

$$= \lim_{m\to\infty} \sum_{n=-m}^{m-1} \left\langle f(u), K_{\phi}\left(u, \sigma + i\tau_{n,m}\right) \right\rangle \psi\left(\sigma + i\tau_{n,m}\right) \Delta\tau_{n,m} \quad \ldots(2.2)$$

$$= \lim_{m\to\infty} \left\langle f(u), \sum_{n=-m}^{m-1} K_{\phi}\left(u, \sigma + i\tau_{n,m}\right) \psi\left(\sigma + i\tau_{n,m}\right) \Delta\tau_{n,m} \right\rangle$$

We show that the last summation converges in E to the integral on the right hand side of (2.1). Carrying the operator D_u^k within the integral and summation sign, which is easily justified we get,

$$\gamma_{K,k}\left\{ \sum_{n=-m}^{m-1} K_{\phi}\left(u, \sigma + i\tau_{n,m}\right) \psi\left(\sigma + i\tau_{n,m}\right) \Delta\tau_{n,m} - \int_{-r}^{r} \psi(\varsigma) K_{\phi}(u,\varsigma)\, d\tau \right\}$$

$$= \sup_{u\in K} \left\{ \sum_{n=-m}^{m-1} D_u^k K_{\phi}\left(u, \sigma + i\tau_{n,m}\right) \psi\left(\sigma + i\tau_{n,m}\right) \Delta\tau_{n,m} - \int_{-r}^{r} \psi(\varsigma) K_{\phi}(u,\varsigma)\, d\tau \right\}.$$

As $\lim_{m\to\infty} \sum_{n=-m}^{m-1} D_u^k K_{\phi}\left(u, \sigma + i\tau_{n,m}\right) \psi\left(\sigma + i\tau_{n,m}\right) \Delta\tau_{n,m} = \int_{-r}^{r} \psi(\varsigma) K_{\phi}(u,\varsigma)\, d\tau, \quad \forall u \in K.$

It thus follows that for every m, the summation is a member of E and it converges in E to the integral on the right hand side of (2.1). Hence the proof.

Lemma 2: For $\theta(x)\in \mathcal{E}$, set $\psi(\varsigma)$ as in lemma 3.7.3 above for $\varsigma\in C$, u is restricted to a compact subset of R then,

$$M_r(u)=\frac{1}{\pi}\int_{-r}^{r}K_\phi(u,\varsigma)\,\psi(\varsigma)d\tau$$

$$=\frac{1}{\pi}\int_{-r}^{r}K_\phi(u,\varsigma)\int_{-\infty}^{\infty}\theta(x)\overline{K_\phi(x,\varsigma)}\,dx\,d\tau \qquad \ldots(2.3)$$

converges in E to $\theta(u)$ as $r\to\infty$.

Proof : We shall show that $M_r(u)\to\theta(u)$ in E as $r\to\infty$.

That is to show,

$$\gamma_{K,k}\left[M_r(u)-\theta(u)\right]=\underset{u\in K}{Sup}\left\{D_u^k\left[M_r(u)-\theta(u)\right]\right\}\to 0 \text{ as r}\to\infty.$$

We note that for k = 0,

$$\frac{1}{\pi}\int_{0}^{\infty}K_\phi(u,\varsigma)\int_{-\infty}^{\infty}\theta(x)\overline{K_\phi(x,\varsigma)}\,dx\,d\tau=\theta(u)$$

That is to say that $\lim_{r\to\infty}M_r(u)=\theta(u)$

Since the integrand is a C^∞ - function of u and $\theta\in$ E, we can repeatedly differentiate under the integral sign and the integrals are uniformly convergent we have

$$\frac{1}{\pi}\int_{0}^{\infty}D_u^kK_\phi(u,\varsigma)\int_{-\infty}^{\infty}\theta(x)\overline{K_\phi(x,\varsigma)}\,dx\,d\tau=\theta(u) \text{ for all } u\in K$$

Hence the claim.

Proof of inversion theorem

Now let $\theta(x)\in\mathcal{E}$. We shall show that

$$\left\langle\frac{1}{\pi}\int_{-r}^{r}\overline{K_\phi(x,\varsigma)}M_\phi(\varsigma)\,d\tau\ ,\quad\theta(x)\right\rangle \text{ tends to } \left\langle f(x),\theta(x)\right\rangle \text{ as r}\to\infty. \qquad \ldots(2.4)$$

From the analyticity of $M_\phi(\varsigma)$ on C and the fact that $\psi(x)$ has a compact support in R, it follows that the left side expression in (2.4) is merely a repeated integral with respect to x and ς and the integral in (2.4) is a continuous function of x as the closed bounded domain of the integration.

Therefore we write (2.4) as -

$$\frac{1}{\pi}\int_{0}^{\infty}\theta(x)\int_{-r}^{r}\overline{K_\phi(x,\varsigma)}M_\phi(\varsigma)\,d\tau\,dx=\frac{1}{\pi}\int_{0}^{\infty}\theta(x)\int_{-r}^{r}\overline{K_\phi(x,\varsigma)}\left\langle f(u),K_\phi(u,\varsigma)\right\rangle d\tau\,dx$$

$$= \frac{1}{\pi} \int_{-r}^{r} \left\langle f(u), K_{\phi}(u,\varsigma) \right\rangle \int_{0}^{\infty} \theta(x) \overline{K_{\phi}(x,\varsigma)} \, dx \, d\tau$$

Since $\theta(x)$ is of compact support, and the integrand is a continuous function of (x,ς) the order of integration may be changed. The change in the order of integration is justified by appeal to lemma 1.

Thisyields $\frac{1}{\pi} \int_{0}^{\infty} \theta(x) \int_{-r}^{r} \overline{K_{\phi}(x,\varsigma)} M_{\phi}(\varsigma) \, d\tau \, dx = \frac{1}{\pi} \int_{-r}^{r} \left\langle f(u), K_{\phi}(u,\varsigma) \right\rangle \psi(\varsigma) \, d\tau$,

where $\psi(\varsigma)$ is as in lemma 1.

This is equal to $\left\langle f(u), \frac{1}{\pi} \int_{-r}^{r} K_{\phi}(u,\varsigma) \psi(\varsigma) d\tau \right\rangle$ …(2.5)

Again by lemma 2 equation (2.5) converges to $\left\langle f(u), \theta(u) \right\rangle$ as r $\to \infty$.

This completes the proof of the theorem.

Application of fractional mellin transform

Scale transform is a powerful mathematical tool for processing images (for detecting) that are arbitrarily scaled. Hence it is used in the class of linear streach invariant systems. Xiaohong Hu[8] developed Mellin transform technique of probability modeling for accurate solution of problems in some industrial statistic.

Fractional Mellin transform given by Akay adds one more parameter (angle) to scale transform and hence it is also used in pattern recognition problems, industrial statistic. Moreover fractional Mellin based correlators are used to obtain time to impact and controlling moments in the navigation task[7].

The generalized fractional Mellin transform we have introduced in this paper is the extension of fractional Mellin transform given by Akay and can be used in all above cases. The advantage of our generalized fractional Mellin transform is it can be used even when the signals (functions) are singular functions.

CONCLUSION

We have given the inversion theorem for the generalized fractional complex Melin transform with two lemmas. Given some applications in various fields of this generalized fractional complex Mellin transform.

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