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A COMMON FIXED POINT THEOREM FOR SOFT (α,β) - ψ-CONTRACTIVE TYPE MAPPINGS WITH APPLICATIONS

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ABSTRACT

In this paper, new concept of soft (α, β) - ψ - contractive mappings have been introduced and we obtain a unique common fixed point theorem in soft metric space has been obtained. An example and application has been given to support these results.

Key words: Soft metric, Soft element, Soft (α, β) - ψ - contraction.

INTRODUCTION

In the year 1999, Molodtsov[1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Maji et al.[2] introduced several operations on soft sets and applied it to decision making problems. Then the idea of soft topological space was first given by Shabir and Naz[3]. Das and Samanta[4] introduced the notion of soft metric space and investigated some basic properties of this space. In 2012, Samet et al.[5] introduced α - ψ contractive mappings and gave some results on a fixed point of the mappings. By using their main idea a new concept of soft (α,β) - ψ - contractive mappings was introduced and a unique common fixed point theorem in soft metric space was obtained. An example and application has been given to support our results.

Definition 1.1[2] Let X be an initial universe set and E be a set of parameters. A pair (F,E) is called a soft set over X, if and only if F is a mapping from E into the set of all subsets of the set X, i.e.,

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F : E → P(X), where P(X) is the power set of X.

Definition 1.2[6] A soft set (F,E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all e ∈ E, F(e) = X.

Definition 1.3[7] Let R be the set of real numbers and B(R) be the collection of all non-empty bounded subsets of R and E be taken as a set of parameters. Then a mapping F : E → B(R) is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\widetilde{r}, \widetilde{t}, \widetilde{s}$ etc. $\widetilde{0}$ and $\widetilde{1}$ are the soft real numbers where $\widetilde{0}$(e) = 0, $\widetilde{1}$(e) = 1 for all e ∈ E, respectively.

Definition 1.4. (Soft element). Let X be a non-empty set and E be a non-empty parameter set. Then a function $\boldsymbol{\varepsilon}$: E → X is said to be a soft element of X. A soft element ε of X is said to belong to a soft set A of X, denoted by $\boldsymbol{\varepsilon}$ ∈ A, if $\boldsymbol{\varepsilon}$(e) ∈ A(e), e ∈ E. Thus a soft set A of X with respect to the index set E can be expressed as A(e) = {$\boldsymbol{\varepsilon}$ (e), $\boldsymbol{\varepsilon}$ ∈ A}, e ∈ E.

Note. It is to be noted that every singleton soft set (a soft set (F; A) for which F(e) is a singleton set, $\forall \boldsymbol{\lambda} \in$ A) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall \boldsymbol{\lambda} \in$ A.

Definition 1.5[7] Let $\widetilde{r}, \widetilde{s}$ be two soft real numbers. Then the following statements.

hold:

(i) $\widetilde{r} \,\widetilde{\leq}\, \widetilde{s}$ if $\widetilde{r}(e) \,\widetilde{\leq}\, \widetilde{s}(e)$ for all e ∈ E;

(ii) $\widetilde{r} \,\widetilde{\geq}\, \widetilde{s}$ if $\widetilde{r}(e) \,\widetilde{\geq}\, \widetilde{s}(e)$ for all e ∈ E;

(iii) $\widetilde{r} \,\widetilde{<}\, \widetilde{s}$ if $\widetilde{r}(e) \,\widetilde{<}\, \widetilde{s}(e)$ for all e ∈ E;

(iv $\widetilde{r} \,\widetilde{>}\, \widetilde{s}$ if $\widetilde{r} \,\widetilde{>}\, \widetilde{s}$ for all e ∈ E:

Let SE ($\widetilde{X}$) be the collection of all soft points of $\widetilde{X}$ and R(E)* denote the set of all non-negative soft real numbers.

Definition 1.6[4] A mapping d : SE ($\widetilde{X}$) × SE ($\widetilde{X}$) → R(E)*, is said to be a soft metric on the soft set $\widetilde{X}$ if d satisfies the following conditions:

(M1) $d(\tilde{x}, \tilde{y}) \tilde{\geq} \bar{0} \ \forall \ \tilde{x}, \tilde{y} \in \tilde{X}$;

(M2) $d(\tilde{x}, \tilde{y}) = \bar{0}$ iff $\tilde{x} = \tilde{y}$;

(M3) $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x}) \forall \tilde{x}, \tilde{y} \in \tilde{X}$;

(M4) For all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$; $d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$

The soft set $\tilde{X}$ with a soft metric d on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, d, E)$.

Theorem 1.7[4] (Decomposition Theorem) If a soft metric d satisfies the condition:

(M5): For $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda): \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \lambda\}$ is a singleton set, and if for $\lambda \in A$, $d_\lambda : X \times X \to R^+$ is defined by $d_\lambda(\tilde{x}(\lambda), \tilde{y}(\lambda)) = d(\tilde{x}, \tilde{y})(\lambda)$, $\tilde{x}, \tilde{y} \in \tilde{X}$, then d_λ is a metric on $\tilde{X}$.

Definition 1.8[4] Let $(\tilde{X}, d, E)$ be a soft metric space $\tilde{\varepsilon}$ be a non-negative soft real number.

$B(\tilde{x}, \tilde{\varepsilon}) = \{\tilde{y} \in \tilde{X} : \tilde{d}(\tilde{x}, \tilde{y}) \tilde{<} \tilde{\varepsilon}\} \subseteq SE(\tilde{X})$ is called the soft open ball with center at $\tilde{x}$ and radius $\tilde{\varepsilon}$ and $B[\tilde{x}, \tilde{\varepsilon}] = \{\tilde{y} \in \tilde{X} : \tilde{d}(\tilde{x}, \tilde{y}) \tilde{\leq} \tilde{\varepsilon}\} \subseteq SE(\tilde{X})$ is called the soft closed ball with center at $\tilde{x}$ and radius $\tilde{\varepsilon}$.

Definition 1.9[4] Let $\{\tilde{x}_n\}$ be a sequence of soft elements in a soft metric space $(\tilde{X}, d, E)$. The sequence $\{\tilde{x}_n\}$ is said to be convergent in $(\tilde{X}, d, E)$ if there is a soft element $\tilde{y} \in \tilde{X}$ such that $d(\tilde{x}_n, \tilde{y}) \to \bar{0}$ as $n \to \infty$. This means for every $\tilde{\varepsilon} \tilde{\geq} \bar{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\tilde{\varepsilon})$ such that $\bar{0} \tilde{\leq} d(\tilde{x}_n, \tilde{y}) \tilde{<} \tilde{\varepsilon}$, whenever $n > N$.

Theorem 1.10[4] Limit of a sequence in a soft metric space, if exist, is unique.

Definition 1.11[4] A sequence $\{\tilde{x}_n\}$ of soft elements in $(\tilde{X}, d, E)$ is considered as a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\varepsilon} > \bar{0}$, $\exists$ $m \in N$ such that $\tilde{d}(\tilde{x}_i, \tilde{x}_j) \tilde{\leq} \tilde{\varepsilon}, \forall \ i, j \geq m$.

Definition 1.12[4] A soft metric space $(\tilde{X}, d, E)$ is called complete if every Cauchy

Sequence in $\tilde{X}$ converges to some point of $\tilde{X}$. The soft metric space ($\tilde{X}$, d, E) is called incomplete if it is not complete.

Definition 1.13[4] Let ($\tilde{X}$, d, E) be a soft metric space. We can consider X as the collection of all soft elements of X with respct to a parameter set A. Let f : ($\tilde{X}$, d) → ($\tilde{X}$, d) be a mapping. If there exists a soft element $\tilde{x}_0 \in \tilde{X}$ such that f($\tilde{x}_0$) = $\tilde{x}_0$, then $\tilde{x}_0$ is called a fixed element of f.

Definition 1.14[4] Let ($\tilde{X}$, d) be a soft metric space. We can consider $\tilde{X}$ as the collection of all soft elements of $\tilde{X}$ with respect to a parameter set A. A mapping

f: ($\tilde{X}$, d) → ($\tilde{X}$, d) is said to be a contraction mapping in ($\tilde{X}$, d), if there is positive soft real number $\tilde{t}$ with $\bar{0} < \tilde{t} < \bar{1}$ such that $d(f(\tilde{x}), f(\tilde{y})) \tilde{\leq} \tilde{t}\, d(\tilde{x}, \tilde{y})$, $\tilde{x}, \tilde{y} \in \tilde{X}$

Theorem 1.15[4] Let ($\tilde{X}$, d) be a complete soft metric space. Let f : ($\tilde{X}$, d) → ($\tilde{X}$, d) be a contraction mapping. Then f has a unique fixed element.

Definition 1.16. Let Ψ be the family of functions ψ: $R(E)^* \to R(E)^*$ satisfying the following conditions.

(i) ψ is non-decreasing

(ii) $\sum_{n=1}^{\infty} \psi^n(\tilde{t}) < \infty$ for all $\tilde{t} > \bar{0}$, where ψ^n is the nth iterative of ψ .

Remark: For every function ψ : $R(E)^* \to R(E)^*$ the following holds:

if ψ is non decreasing, then for each $\tilde{t} > \bar{0}$, $\lim_{n\to\infty} \psi^n(\tilde{t}) = \bar{0} \Rightarrow \psi(\tilde{t}) < \tilde{t} \Rightarrow \psi(\bar{0}) = \bar{0}$.

There fore if ψ ∈ Ψ then for each $\tilde{t} > \bar{0}$, $\psi(\tilde{t}) < \tilde{t} \Rightarrow \psi(\bar{0}) = \bar{0}$.

The notations F(f,T) and C(f, T) stand for the set of all common fixed point and and the set of all coincidence points of f and T, respectively.

Definition 1.17[5] Let T: X → X and α: X × X → [0, ∞) we say that T is α-admissible if x; y ∈ X; α (x, y) ≥ 1 ⇒ α (Tx, Ty) ≥ 1.

Definition 1.18. Let T : $\widetilde{X} \to \widetilde{X}$ and α, β : $\widetilde{X} \times \widetilde{X} \to R(E)^*$ we say that T is soft (α,β) - admissible if $\widetilde{x}, \widetilde{y} \in \widetilde{X}$, $\alpha(\widetilde{x}, \widetilde{y}) \widetilde{\geq} \overline{1}$, $\beta(\widetilde{x}, \widetilde{y}) \widetilde{\geq} \overline{1} \Rightarrow \alpha(T\widetilde{x}, T\widetilde{y}) \widetilde{\geq} \overline{1}$, $\beta(T\widetilde{x}, T\widetilde{y}) \widetilde{\geq} \overline{1}$.

Definition 1.19. Let f, g, S, T : X → X be four self mappings of a nonempty set X, and let α: S(X) ∪ T(X) × S(X) ∪ T(X) → $[0, \infty)$ be mappings , then the pair (f, g) is called an α-admissible with respect toS and T (in short α(S,T) - admissible)if for all x, y ∈X α (Sx, Ty) ≥ 1 or α (Tx, Sy) ≥ 1 implies α (fx, gy) ≥ 1 and α (gx, fy) ≥ 1.

Definition 1.20. Let f, g, S, T : $\widetilde{X} \to \widetilde{X}$ be four self mappings of a nonempty set $\widetilde{X}$, and let α,β: f(X) ∪ g(X) × f(X) ∪ g(X) → $[0, \infty)$ be mappings, then the pair (S,T) is called an soft (α,β)- admissible with respect to f and g (in short (S,T) is soft $(\boldsymbol{\alpha}, \boldsymbol{\beta})_{(f,g)}$ - admissible) if for all $\widetilde{x}, \widetilde{y} \in \widetilde{X}$, $\alpha(f\widetilde{x}, g\widetilde{y}) \geq \overline{1}$, $\beta(f\widetilde{x}, g\widetilde{y}) \geq \overline{1}$ or $\alpha(g\widetilde{x}, f\widetilde{y}) \geq \overline{1}$, $\beta(g\widetilde{x}, f\widetilde{y}) \geq \overline{1}$ implies $\alpha(S\widetilde{x}, T\widetilde{y}) \geq \overline{1}$, $\beta(S\widetilde{x}, T\widetilde{y}) \geq \overline{1}$ or $\alpha(T\widetilde{x}, S\widetilde{y}) \geq \overline{1}$, $\beta(T\widetilde{x}, S\widetilde{y}) \geq \overline{1}$.

Definition 1.21. Let ($\widetilde{X}$, d) be a soft metric space and f, g, S, T : $\widetilde{X} \to \widetilde{X}$ be amappings and (S, T) be $(\boldsymbol{\alpha}, \boldsymbol{\beta})_{(f,g)}$ - admissible pair, we say that (S, T) is soft $(\boldsymbol{\alpha}, \boldsymbol{\beta})_{(f,g)}$ - contraction if

$$\boldsymbol{\alpha}(f\widetilde{x}, g\widetilde{y})\boldsymbol{\beta}(f\widetilde{x}, g\widetilde{y})d(S\widetilde{x}, T\widetilde{y}) \leq \boldsymbol{\psi}(M(\widetilde{x}, \widetilde{y})) \qquad \ldots(1)$$

Where $M(\widetilde{x}, \widetilde{y}) = \max\left\{\begin{matrix} d(f\widetilde{x}, g\widetilde{y}), \quad d(f\widetilde{x}, S\widetilde{x}), \quad d(g\widetilde{y}, T\widetilde{y}), \\ \frac{1}{2}\left[d(f\widetilde{x}, T\widetilde{y}) + d(g\widetilde{y}, S\widetilde{x})\right] \end{matrix}\right\}$

for all $\widetilde{x}, \widetilde{y} \in \widetilde{X}$, and $\psi \in \Psi$.

Definition 1.22[8] Let ($\widetilde{X}$, d , E) be a soft metric space and two mappings f, g : ($\widetilde{X}$, d, E) → ($\widetilde{X}$, d, E) are said to be soft weakly compatible if $f(g(\widetilde{x})) = g(f(\widetilde{x}))$ for all $\widetilde{x} \in \widetilde{X}$, which satisfy $f(\widetilde{x}) = g(\widetilde{x})$.

Now we prove our main result.

Main result

Theorem 2.1. Let f, g, S, T: $\widetilde{X} \to \widetilde{X}$ be a mappings on a complete soft metric space with $S(\widetilde{X}) \subseteq g(\widetilde{X})$, $T(\widetilde{X}) \subseteq f(\widetilde{X})$ and (S, T) be soft $(\boldsymbol{\alpha}, \boldsymbol{\beta})_{(f,g)}$ - contractive pair.

Suppose that the following conditions are satisfied:

(2.1.1) there exists $\tilde{x}_0 \in \tilde{X}$ such that $\alpha(f\tilde{x}_0, g\tilde{x}_0) \geq \bar{1}$ and $\beta(f\tilde{x}_0, g\tilde{x}_0) \geq \bar{1}$

(2.1.2) $\alpha(f\tilde{x}_n, g\tilde{x}_{n+1}) \geq \bar{1}$, $\beta(f\tilde{x}_n, g\tilde{x}_{n+1}) \geq \bar{1}$, $\forall$ n even implies that $\alpha(f\tilde{x}_n, g\tilde{x}_j) \geq \bar{1}$, $\beta(f\tilde{x}_n, g\tilde{x}_j) \geq \bar{1}$,

(2.1.3) $\alpha(f\tilde{x}_n, g\tilde{x}_{n+1}) \geq \bar{1}$, $\beta(f\tilde{x}_n, g\tilde{x}_{n+1}) \geq \bar{1}$, $\forall$ n even and $f\tilde{x}_n$ and $g\tilde{x}_{n+1}$ converges to $\tilde{x} \in \tilde{X}$ as $n \to \infty$ implies that $\alpha(f\tilde{x}_n, \tilde{x}) \geq \bar{1}$, $\beta(f\tilde{x}_n, \tilde{x}) \geq \bar{1}$, and $\alpha(\tilde{x}, g\tilde{x}_{n+1}) \geq \bar{1}$, $\beta(\tilde{x}, g\tilde{x}_{n+1}) \geq \bar{1}$ $\forall$ n even .

Then the pairs (S, f) and (T, g) have a point of coincidence in X. Moreover if

(i) (S, f) and (T, g) are soft weakly compatiable

(ii) $\alpha(f\tilde{u}, g\tilde{v}) \geq \bar{1}$ and $\beta(f\tilde{u}, g\tilde{v}) \geq \bar{1}$, whenever $\tilde{u} \in C(S, f)$ and $\tilde{v} \in C(T, g)$.

Then f, g, S and T has a common fixed soft element.

Proof. Let $\tilde{x}_0 \in \tilde{X}$ such that $\alpha(f\tilde{x}_0, g\tilde{x}_0) \geq \bar{1}$ and $\beta(f\tilde{x}_0, g\tilde{x}_0) \geq \bar{1}$. Since $S(\tilde{X}) \subseteq g(\tilde{X})$, there exists $\tilde{x}_1 \in \tilde{X}$ such that $S\tilde{x}_0 = g\tilde{x}_1$. Again since $T(\tilde{X}) \subseteq f(\tilde{X})$, there exists $\tilde{x}_2 \in \tilde{X}$ such that $T\tilde{x}_1 = f\tilde{x}_2$. Continuing this process, we can construct the sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ in $\tilde{X}$ defined by –

$$\tilde{y}_{2n} = S\tilde{x}_{2n} = g\tilde{x}_{2n+1}, \quad \tilde{y}_{2n+1} = f\tilde{x}_{2n+2} = T\tilde{x}_{2n+1}, \text{ n} = 0,1,2\cdots \qquad \ldots(2)$$

Since (S, T) is soft $(\alpha, \beta)_{(f,g)}$ - admissible pair

$\alpha(f\tilde{x}_0, S\tilde{x}_0) = \alpha(f\tilde{x}_0, g\tilde{x}_1) \geq \bar{1} \Rightarrow \alpha(S\tilde{x}_0, T\tilde{x}_1) \geq \bar{1} = \alpha(g\tilde{x}_1, f\tilde{x}_2) \geq \bar{1}$

$\alpha(g\tilde{x}_1, f\tilde{x}_2) \geq \bar{1} \Rightarrow \alpha(S\tilde{x}_1, T\tilde{x}_2) \geq \bar{1}$ and $\alpha(T\tilde{x}_1, S\tilde{x}_2) \geq \bar{1}$

which gives $\alpha(f\tilde{x}_2, g\tilde{x}_3) \geq \bar{1}$ continuing this way, we obtain

$\alpha(f\tilde{x}_{2n}, g\tilde{x}_{2n+1}) \geq \bar{1}$ similarly $\beta(f\tilde{x}_{2n}, g\tilde{x}_{2n+1}) \geq \bar{1}$. Therefore

$$\alpha(f\tilde{x}_{2n}, g\tilde{x}_{2n+1}) \geq \bar{1} \text{ and } \beta(f\tilde{x}_{2n}, g\tilde{x}_{2n+1}) \geq \bar{1} \qquad \ldots(3)$$

putting $\tilde{x} = \tilde{x}_{2n}$ and $\tilde{y} = \tilde{x}_{2n+1}$ in (1) and using (2), (3)

$$\begin{aligned} d(\tilde{y}_{2n}, \tilde{y}_{2n+1}) &= d(S\tilde{x}_{2n}, T\tilde{x}_{2n+1}) \\ &\leq \boldsymbol{\alpha}(f\tilde{x}_{2n}, g\tilde{x}_{2n+1})\boldsymbol{\beta}(f\tilde{x}_{2n}, g\tilde{x}_{2n+1}) d(S\tilde{x}_{2n}, T\tilde{x}_{2n+1}) \\ &\leq \boldsymbol{\psi}(M(\tilde{x}_{2n}, \tilde{x}_{2n+1})) \end{aligned}$$

Where

$$\begin{aligned} M(\tilde{x}_{2n}, \tilde{x}_{n+1}) &= \max\left\{ \begin{matrix} d(\tilde{y}_{2n-1}, \tilde{y}_{2n}), \quad d(\tilde{y}_{2n-1}, \tilde{y}_{2n}), \quad d(\tilde{y}_{2n}, \tilde{y}_{2n+1}), \\ \frac{1}{2}\left[d(\tilde{y}_{2n-1}, \tilde{y}_{2n+1}) + d(\tilde{y}_{2n}, \tilde{y}_{2n})\right] \end{matrix} \right\} \\ &= \max\{d(\tilde{y}_{2n-1}, \tilde{y}_{2n}), \quad d(\tilde{y}_{2n}, \tilde{y}_{2n+1})\}. \end{aligned}$$

Thus

$$d(\tilde{y}_{2n}, \tilde{y}_{2n+1}) \leq \boldsymbol{\psi}(\max\{d(\tilde{y}_{2n-1}, \tilde{y}_{2n}), \quad d(\tilde{y}_{2n}, \tilde{y}_{2n+1})\}).$$

If $d(\tilde{y}_{2n}, \tilde{y}_{2n+1})$ is maximum, then

$$d(\tilde{y}_{2n}, \tilde{y}_{2n+1}) \leq \boldsymbol{\psi}(d(\tilde{y}_{2n}, \tilde{y}_{2n+1})) < d(\tilde{y}_{2n}, \tilde{y}_{2n+1})$$

is a contradiction.

Hence

$$d(\tilde{y}_{2n}, \tilde{y}_{2n+1}) \leq \boldsymbol{\psi}(d(\tilde{y}_{2n-1}, \tilde{y}_{2n})).$$

Continuing this way, we get –

$$d(\tilde{y}_{2n}, \tilde{y}_{2n+1}) \leq \boldsymbol{\psi}^n(d(\tilde{y}_0, \tilde{y}_1)) \qquad \ldots(4)$$

From (4) and using the triangular inequality, for all n $\in$ N.

$$d(\tilde{y}_n, \tilde{y}_{n+1}) \leq \boldsymbol{\psi}^n(d(\tilde{y}_0, \tilde{y}_1)).$$

Therefore, for all n,m $\in$ N, n < m, by the triangle inequality we obtain –

$$d(\tilde{y}_n,\tilde{y}_m) \le d(\tilde{y}_n,\tilde{y}_{n+1}) + d(\tilde{y}_{n+1},\tilde{y}_{n+2}) + \cdots + d(\tilde{y}_{m-1},\tilde{y}_m)$$
$$\le \sum_{p=n}^{m-1} \boldsymbol{\psi}^p(d(\tilde{y}_0,\tilde{y}_1))$$
$$\le \sum_{p=n}^{\infty} \boldsymbol{\psi}^p(d(\tilde{y}_0,\tilde{y}_1)).$$

Letting p $\to\infty$, we obtain that $\{\tilde{y}_n\}$ is a Cauchy sequence in ($\tilde{X}$, d) and hence $\{\tilde{y}_{2n}\}$ is a Cauchy sequence. Since ($\tilde{X}$, d) be a complete soft metric space. There exists $\tilde{x} \in \tilde{X}$ such that –

$$\lim_{n\to\infty} \tilde{y}_n = \tilde{x}. \qquad \ldots(5)$$

From (2) and (5), we get –

$$S\tilde{x}_{2n} \to \tilde{x}, g\tilde{x}_{2n+1} \to \tilde{x}, \quad f\tilde{x}_{2n+2} \to \tilde{x}, T\tilde{x}_{2n+1} \to \tilde{x} \text{ as } \ n \to \infty \qquad \ldots(6)$$

Now we shall prove that $\tilde{x}$ is a common fixed soft element of f, g, S and T.

Since T($\tilde{X}$) $\subseteq$ f($\tilde{X}$), we can choose a soft element $\tilde{u} \in \tilde{X}$ such that $\tilde{x} = f\tilde{u}$

Suppose that d($\tilde{x}, S\tilde{u}$) $\neq \overline{0}$.

By using (3), (6) and (2.1.3), we have

α($\tilde{u}$,g $\tilde{x}_{n+1}$) $\geq \overline{1}$, β ($\tilde{u}$,g $\tilde{x}_{n+1}$) $\geq \overline{1}$

Then substituting $\tilde{x} = \tilde{u}$ and $\tilde{y} = \tilde{x}_{n+1}$ in (1), we get –

$$d(S\tilde{u},T\tilde{x}_{2n+1}) \le \boldsymbol{\alpha}(f\tilde{u},g\tilde{x}_{2n+1})\boldsymbol{\beta}(f\tilde{u},g\tilde{x}_{2n+1})d(S\tilde{u},T\tilde{x}_{2n+1})$$
$$\le \boldsymbol{\psi}(M(\tilde{u},\tilde{x}_{2n+1}))$$
$$= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(f\tilde{u},g\tilde{x}_{2n+1}), \quad d(f\tilde{u},S\tilde{u}), \quad d(g\tilde{x}_{2n+1},T\tilde{x}_{2n+1}), \\ \frac{1}{2}\left[d(f\tilde{u},T\tilde{x}_{2n+1}) + d(g\tilde{x}_{2n+1},S\tilde{u})\right] \end{matrix}\right\}\right).$$

Letting n $\to\infty$, we get –

$$d(S\tilde{u},\tilde{x}) \leq \psi\left(\max\left\{\begin{matrix}\overline{0}, & d(\tilde{x},S\tilde{u}), & \overline{0}, \\ \frac{1}{2}\left[\overline{0}+d(\tilde{x},S\tilde{u})\right] & & \end{matrix}\right\}\right)$$
$$\leq \psi(d(\tilde{x},S\tilde{u}))$$
$$= \psi(d(S\tilde{u},\tilde{x})) < d(S\tilde{u},\tilde{x})$$

is a contradiction.

Hence f $\tilde{u}$ = S $\tilde{u}$ = $\tilde{x}$, and so $\tilde{u}$ ∈ C(S, f).

Similarly, since S($\tilde{X}$) $\subseteq$ g($\tilde{X}$), we can choose a soft element $\tilde{v} \in \tilde{X}$ such that $\tilde{x} = g\tilde{v}$.

Suppose that d ($\tilde{x}$, T $\tilde{v}$) $\neq \overline{0}$.

By using (3), (6) and (2.1.3), we have –

α(f $\tilde{x}_{2n}$,g $\tilde{v}$) ≥ $\overline{1}$ and β (f $\tilde{x}_{2n}$, $\tilde{v}$) ≥ $\overline{1}$

Then substituting $\tilde{x}$ = $\tilde{x}_{2n}$ and $\tilde{y}$ = $\tilde{v}$ in (1), we get –

$$d(S\tilde{x}_{2n},T\tilde{v}) \leq \boldsymbol{\alpha}(f\tilde{x}_{2n},g\tilde{v}\boldsymbol{\beta}(f\tilde{x}_{2n},g\tilde{v})d(S\tilde{x}_{2n},T\tilde{v})$$
$$\leq \boldsymbol{\psi}(M(\tilde{x}_{2n},\tilde{v}))$$
$$= \boldsymbol{\psi}\left(\max\left\{\begin{matrix}d(f\tilde{x}_{2n},g\tilde{v}), & d(f\tilde{x}_{2n},S\tilde{x}_{2n}), & d(g\tilde{v},T\tilde{v}), \\ \frac{1}{2}\left[d(f\tilde{x}_{2n},T\tilde{v})+d(g\tilde{v},S\tilde{x}_{2n})\right] & & \end{matrix}\right\}\right).$$

Letting n $\to \infty$, we get –

$$d(\tilde{x},T\tilde{v}) \leq \boldsymbol{\psi}\left(\max\left\{\begin{matrix}\overline{0}, & d(\tilde{x},T\tilde{v}), & \overline{0}, \\ \frac{1}{2}\left[\overline{0}+d(\tilde{x},T\tilde{v})\right] & & \end{matrix}\right\}\right)$$
$$\leq \boldsymbol{\psi}(d(\tilde{x},T\tilde{v})) < d(\tilde{x},T\tilde{v})$$

is a contradiction.

Hence g $\tilde{v}$ = T $\tilde{v}$ = $\tilde{x}$, and so $\tilde{v}$ ∈ C(T, g).

Therefore $\tilde{x} = g\tilde{v} = T\tilde{v} = f\tilde{u} = S\tilde{u}$. By the weak compatibility of the pairs (S, f) and (T,g) we deduce that $f\tilde{x} = S\tilde{x}$ and $g\tilde{x} = T\tilde{x}$. Since $\tilde{x} \in C(S, f)$ and $\tilde{v} \in C(g, T)$ by (ii), we have $\alpha(f\tilde{x}, g\tilde{v}) \geq \bar{1}$ and $\beta(f\tilde{x}, g\tilde{v}) \geq \bar{1}$.

From (2.1.1), we get –

$$\begin{aligned} d(S\tilde{x},\tilde{x}) = d(S\tilde{x},T\tilde{v}) &\leq \boldsymbol{\alpha}(f\tilde{x},g\tilde{v})\boldsymbol{\beta}(f\tilde{x},g\tilde{v})d(S\tilde{x},T\tilde{v}) \\ &\leq \boldsymbol{\psi}(M(\tilde{x},\tilde{v})) \\ &= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(f\tilde{x},g\tilde{v}), \quad d(f\tilde{x},S\tilde{x}), \quad d(g\tilde{v},T\tilde{v}), \\ \frac{1}{2}[d(f\tilde{x},T\tilde{v}) + d(g\tilde{v},S\tilde{x})] \end{matrix}\right\}\right) \\ &= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(f\tilde{x},\tilde{x}), \quad \bar{0}, \quad \bar{0}, \\ \frac{1}{2}[d(f\tilde{x},\tilde{x}) + \bar{0}] \end{matrix}\right\}\right) \\ &= \boldsymbol{\psi}(d(f\tilde{x},\tilde{x})) = \boldsymbol{\psi}(d(S\tilde{x},\tilde{x})) \\ &< d(S\tilde{x},\tilde{x}) \end{aligned}$$

is a contradiction.

Hence $f\tilde{x} = S\tilde{x} = \tilde{x}$.

Similarly, $\tilde{u} \in C(S; f)$ and $\tilde{x} \in C(T, g)$ by (ii), we have

$\alpha(f\tilde{u}, g\tilde{x}) \geq \bar{1}$ and $\beta(f\tilde{u}, g\tilde{x}) \geq \bar{1}$.

From (2.1.1), we get –

$$\begin{aligned} d(\tilde{x},T\tilde{x}) = d(S\tilde{u},T\tilde{x}) &\leq \boldsymbol{\alpha}(f\tilde{u},g\tilde{x})\boldsymbol{\beta}(f\tilde{u},g\tilde{x})d(S\tilde{u},T\tilde{x}) \\ &\leq \boldsymbol{\psi}(M(\tilde{u},\tilde{x})) \\ &= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(f\tilde{u},g\tilde{x}), \quad d(f\tilde{u},S\tilde{u}), \quad d(g\tilde{x},T\tilde{x}), \\ \frac{1}{2}[d(f\tilde{u},T\tilde{x}) + d(g\tilde{x},S\tilde{u})] \end{matrix}\right\}\right) \\ &= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(\tilde{x},g\tilde{x}), \quad \bar{0}, \quad \bar{0}, \\ \frac{1}{2}[d(\tilde{x},T\tilde{x}) + d(g\tilde{x},\tilde{x})] \end{matrix}\right\}\right) \\ &= \boldsymbol{\psi}(d(\tilde{x},T\tilde{x})) < d(\tilde{x},T\tilde{x}) \end{aligned}$$

is a contradiction.

Hence $g\widetilde{x} = T\widetilde{x} = \widetilde{x}$.

Therefore $f\widetilde{x} = S\widetilde{x} = g\widetilde{x} = T\widetilde{x} = \widetilde{x}$.

Hence $\widetilde{x}$ is a common fixed soft element of f, g, S and T.

Let us give the following hypothesis for the uniqueness of the common fixed soft element Theorem 2.1 (H) for all if $\widetilde{x}, \widetilde{y} \in$ F(f, g, S, T), we have

$$\alpha(f\widetilde{x}, g\widetilde{y}) \mathrel{\widetilde{\geq}} \overline{1}, \beta(f\widetilde{x}, g\widetilde{y}) \mathrel{\widetilde{\geq}} \overline{1}$$

Theorem 2.2. Adding condition (H) to the hypothesis of Theorem 2.1, we obtain the uniqueness of the common fixed soft element of f, g, S and T.

Proof. Suppose $f\widetilde{x} = S\widetilde{x} = g\widetilde{x} = T\widetilde{x} = \widetilde{x}$ and $f\widetilde{y} = S\widetilde{y} = g\widetilde{y} = T\widetilde{y} = \widetilde{y}$. Then from (H), we have

$$\alpha(f\widetilde{x}, g\widetilde{y}) \geq \overline{1} \text{ and } \beta(f\widetilde{x}, g\widetilde{y}) \geq \overline{1}.$$

Then applying equation (1) , we get

$$\begin{aligned}
d(S\widetilde{x}, T\widetilde{y}) &\leq \boldsymbol{\alpha}(f\widetilde{x}, g\widetilde{y})\boldsymbol{\beta}(f\widetilde{x}, g\widetilde{y})d(S\widetilde{x}, T\widetilde{y}) \\
&\leq \boldsymbol{\psi}(M(\widetilde{x}, \widetilde{y})) \\
&= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(f\widetilde{x}, g\widetilde{y}), \quad d(f\widetilde{x}, S\widetilde{x}), \quad d(g\widetilde{y}, T\widetilde{y}), \\ \frac{1}{2}\left[d(f\widetilde{x}, T\widetilde{y}) + d(g\widetilde{y}, S\widetilde{x})\right] \end{matrix}\right\}\right) \\
&= \boldsymbol{\psi}\left(\max\left\{\begin{matrix} d(\widetilde{x}, \widetilde{y}), \quad \overline{0}, \quad \overline{0}, \\ \frac{1}{2}\left[d(\widetilde{x}, \widetilde{y}) + d(\widetilde{y}, \widetilde{x})\right] \end{matrix}\right\}\right) \\
&= \boldsymbol{\psi}(d(\widetilde{x}, \widetilde{y})) \\
&< d(\widetilde{x}, \widetilde{y})
\end{aligned}$$

is a contradiction.

Hence $\widetilde{x} = \widetilde{y}$ Therefore $\widetilde{x}$ is a unique common fixed soft element of f, g, S and T.

Example 2.3. Let ($\widetilde{X}$, d, A) be soft metric space. Where X = A = [0, 1] and $d(\widetilde{x}, \widetilde{y}) = |\widetilde{x} - \widetilde{y}|$ for every $\widetilde{x}, \widetilde{y} \in \widetilde{X}$. Let f, g, S, T : $\widetilde{X} \to \widetilde{X}$ be mappings defined by fx = $\frac{x}{8}$, gx = $\frac{x}{6}$, Sx = $\frac{x}{48}$ and Tx = $\frac{x}{64}$. Also we define α, β: f(X) ∪ g(X) × f(X) ∪ g(X) → $[0, \infty)$ by α(x, y) = 1 ∀ x ∈ X and β(x, y) = 1 ∀ x ∈ X, and ψ: R+ → R+ be defined as ψ (t) = $\frac{t}{3}$

Now

$$\begin{aligned}\boldsymbol{\alpha}(\tilde{x},\tilde{y})\boldsymbol{\beta}(\tilde{x},\tilde{y})d(S\tilde{x},T\tilde{y}) = d(S\tilde{x},T\tilde{y})\\ &= \left|\frac{\tilde{x}}{64}-\frac{\tilde{y}}{48}\right|\\ &= \frac{1}{8}\left|\frac{\tilde{x}}{8}-\frac{\tilde{y}}{6}\right|\\ &= \frac{1}{8}d(f\tilde{x},g\tilde{y})\\ &\le \frac{1}{8}M(\tilde{x},\tilde{y})\\ &\le \frac{1}{3}M(\tilde{x},\tilde{y}) = \boldsymbol{\psi}(M(\tilde{x},\tilde{y}))\end{aligned}$$

Therefore all the conditions of Theorem 2.1 are satisfied. $\overline{0}$ is the common fixed soft element for f, g, S and T.

Corollary 2.4. Let ($\tilde{X}$, d) be complete soft metric space and T : $\tilde{X} \to \tilde{X}$ be a mappings and T be soft (α,β)- ψ- contractive mapping. Suppose that the following conditions are satisfied:

(i) There exists $\tilde{x}_0 \in \tilde{X}$ such that α($\tilde{x}_0$,T$\tilde{x}_0$) ≥ $\overline{1}$ and β($\tilde{x}_0$,T$\tilde{x}_0$) ≥ $\overline{1}$

(ii) If $\{\tilde{x}_n\}$ is a sequence in $\tilde{X}$ α ($\tilde{x}_n$, $\tilde{x}_{n+1}$) ≥ $\overline{1}$ and β ($\tilde{x}_n$, $\tilde{x}_{n+1}$) ≥ $\overline{1}$ ∀ n and $\tilde{x}_n$ converges to $\tilde{x} \in \tilde{X}$ as n → ∞, then α ($\tilde{x}_n$, $\tilde{x}$) ≥ $\overline{1}$ and β ($\tilde{x}_n$, $\tilde{x}$) ≥ $\overline{1}$ ∀ n.

Then T has a fixed soft element.

Corollary 2.5. Let ($\tilde{X}$, d) be complete soft metric space and T : $\tilde{X} \to \tilde{X}$ be a mappings and $\boldsymbol{\alpha}(\tilde{x},\tilde{y})\boldsymbol{\beta}(\tilde{x},\tilde{y})d(T\tilde{x},T\tilde{y}) \le \boldsymbol{\psi}(d(\tilde{x},\tilde{y}))$ for all $\tilde{x}$, $\tilde{y} \in \tilde{X}$ and ψ ∈ Ψ. Suppose that the following conditions are satisfied:

(i) There exists $\tilde{x}_0 \in \tilde{X}$ such that $\alpha(\tilde{x}_0, T\tilde{x}_0) \geq \bar{1}$ and $\beta(\tilde{x}_0, T\tilde{x}_0) \geq \bar{1}$

(ii) If $\{\tilde{x}_n\}$ is a sequence in $\tilde{X}$ $\alpha(\tilde{x}_n, \tilde{x}_{n+1}) \geq \bar{1}$ and $\beta(\tilde{x}_n, \tilde{x}_{n+1}) \geq \bar{1}$ $\forall$ n and $\tilde{x}_n$ converges to $\tilde{x} \in \tilde{X}$ as $n \to \infty$, then $\alpha(\tilde{x}_n, \tilde{x}) \geq \bar{1}$ and $\beta(\tilde{x}_n, \tilde{x}) \geq \bar{1}$ $\forall$ n.

Then T has a fixed soft element.

Application to integral equations

An integral equation of the form –

$$\tilde{x}(\lambda)(s) = \tilde{y}(\lambda)(s) + \int_0^1 K(s,t)\tilde{x}(\lambda)(t)dt \qquad \ldots(7)$$

Assume that K(s,t) is continuous. Let $\tilde{y}(\lambda) \in \tilde{C}[0, 1]$. Hence

$$\max_{t\in[0,1]} \int_0^1 k(s,t)dt \leq \frac{1}{2}$$

and $\psi(t) = t$.

We consider first the integral equation on $\tilde{C}[0, 1]$, the space of all continuous functions defined on interval [0, 1] with the metric

$$d(\tilde{x}(\lambda), \tilde{y}(\lambda)) = \max_{t\in[0,1]} \left|\tilde{x}(\lambda)(t) - \tilde{y}(\lambda)(t)\right|$$

write the integral equation (7) in the form $\tilde{x} = T\tilde{x}$, where

$$T\tilde{x}(\lambda)(s) = \tilde{y}(\lambda)(s) + \int_0^1 K(s,t)\tilde{x}(\lambda)(t)dt \qquad \ldots(8)$$

Since k and the function $\tilde{y}$ are continuous, it follows that equation (8) defines an operator

$$T : \tilde{C}[0, 1] \to \tilde{C}[0, 1]$$

It follows that

$$\begin{aligned} d(T\widetilde{x}(\boldsymbol{\lambda}),T\widetilde{y}(\boldsymbol{\lambda})) &= \max_{t\in[0,1]}\left|T\widetilde{x}(\boldsymbol{\lambda})(t)-T\widetilde{y}(\boldsymbol{\lambda})(t)\right| \\ &= \max_{t\in[0,1]}\left|\int_0^1 k(s,t)(\widetilde{x}(\boldsymbol{\lambda})(t)-\widetilde{y}(\boldsymbol{\lambda})(t))dt\right| \\ &\le \max_{t\in[0,1]}\int_0^1 k(s,t)\ \left|\widetilde{x}(\boldsymbol{\lambda})(t)-\widetilde{y}(\boldsymbol{\lambda})(t)\right|dt \\ &\le \max_{u\in[0,1]}\ \left|\widetilde{x}(\boldsymbol{\lambda})(u)-\widetilde{y}(\boldsymbol{\lambda})(u)\right|\max_{t\in[0,1]}\int_0^1 k(s,t)\,dt \\ &\le \frac{1}{2}\max_{u\in[0,1]}\ \left|\widetilde{x}(\boldsymbol{\lambda})(u)-\widetilde{y}(\boldsymbol{\lambda})(u)\right| \\ &= \frac{1}{2}d(\widetilde{x}(\boldsymbol{\lambda}),\widetilde{y}(\boldsymbol{\lambda})) \\ &= \boldsymbol{\psi}(d(\widetilde{x}(\boldsymbol{\lambda}),\widetilde{y}(\boldsymbol{\lambda}))). \end{aligned}$$

Thus, we have

$$d(T\widetilde{x},T\widetilde{y}) \le \boldsymbol{\psi}(d(\widetilde{x},\widetilde{y}))\forall\widetilde{x},\widetilde{y}\in\widetilde{C}[0,1].$$

Lastly, we define $\boldsymbol{\alpha},\boldsymbol{\beta}$: $\widetilde{C}$ [0, 1] × $\widetilde{C}$ [0, 1] →R(E)* by

$$\alpha(\widetilde{x},\widetilde{y}) = \begin{cases} \overline{1}\ if & \widetilde{x},\widetilde{y}\in\widetilde{C}[0,1] \\ \overline{0} & otherwise. \end{cases}$$

and $\boldsymbol{\beta}(\widetilde{x},\widetilde{y}) = \begin{cases} \overline{1}\ if & \widetilde{x},\widetilde{y}\in\widetilde{C}[0,1] \\ \overline{0} & otherwise. \end{cases}$

Then for all $\widetilde{x}$, $\widetilde{y}$ $\in$ $\widetilde{C}[0,1]$,we have

$$\alpha(\widetilde{x},\widetilde{y})\beta(\widetilde{x},\widetilde{y})d(T\widetilde{x},T\widetilde{y}) \le \psi(d(\widetilde{x},\widetilde{y})).$$

Hence hypotheses of Corollary 2.5 are satisfied. Thus, the mapping T has a fixed soft element, which is the solution of integral equation (7).

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