ISSN : 0974 - 7435 Volume 10 Issue 2

BioTechnology
An Indian Journal

FULL PAPER

BTAIJ, 10(2), 2014 [248-252]

Sub-algebras of hilbert algebras in BCK-algebras

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ABSTRACT

The notion of BCK-algebras was formulated first in 1966 by K. Iséki Japanese Mathematician. In this paper we will discuss Sub-algebras of Hilbert Algebras in BCK-algebras and its proposition.

KEYWORDS

BCK-algebra;
Hilbert algebras;
Sub-algebras.

INTRODUCTION

BCK-algebra is originated from two different ways. One of the motivation is based on set theory, another motivation is from classical and non-classical propositional calculi. The notion of ideals in BCK-algebras was introduced by K. Iséki in 1975.The ideal theory plays a fundamental role in the general development of BCK-algebras, Y. L. Liu and J. Meng discussed fuzzy ideal, fuzzy positive implicative and fuzzy implicative ideal in BCI-algebras. We also give a fuzzy ideal of Hilbert Algebras in BCK-algebras, and some propositions. The notion of BCK-algebras was formulated first in 1966 by K. Iséki, Japanese, Mathematician. There are many classes of BCK-algebras, for example, sub-algebras, bounded BCK-algebras, positive implicative BCK-algebra, implicative BCK-algebra, commutative BCK-algebra, BCK-algebras with condition (S), Griss (and semi-Brouwerian) algebras, quasi-commutative BCK-algebras, direct product of BCK-algebras, and so on. Here we will discuss Sub-algebras of Hilbert Algebras in BCK-algebras and its proposition.

Definition

Let H be a set with a binary operation $\to$ and a constant 1. Then $(H,\to,1)$ is called a Hilbert Algebras in BCK-algebra if it satisfies the following conditions:

BCI-1 $(y \to z) \to ((z \to x) \to (y \to x)) = 1$,

BCI-2, $y \to ((y \to x) \to x) = 1$,

BCI-3 $x \to x = 1$,

BCI-4 $y \to x = 1$ **and** $x \to y = 1$ **imply** $x = y$,

BCK-5 $x \to 1 = 1$.

Inwe can define a binary operationby if and only if.

Then is called a Hilbert Algebras in BCK-algebra if it satisfies the following conditions:

BCI- $1'$ $(z \to x) \to (y \to x) \le y \to z$

BCI- $2'$ $(y \to x) \to x \le y$,

BCI- $3'$ $x \le x$,

BCI- $4'$ $x \le y$ **and** $y \le x$ **imply** $x = y$,

BCK- $5'$ $1 \le x$,

BCI- $6'$ $x \le y$ **if and only if** $y \to x = 1$.

Proposition

In a Hilbert Algebras in BCK-algebra $(H,\to,1)$, we have the following properties:

(1) $x \le y$ implies $y \to z \le x \to z$,

(2) $x \le y$ and $y \le z$ imply $x \le z$.

Proof

Let $x \le y$, then by BCI-$1'$, we have

$(x \to z) \to (y \to z) \le y \to x$

Since $x \le y$ implies $y \to x = 1$, we obtain

$(x \to z) \to (y \to z) \le 1$

Combining BCI-$4'$ and BCK-$5'$ we have

$(x \to z) \to (y \to z) = 1$

This is $y \to z \le x \to z$, hence (1) holds.

By (1), $y \le z$ implies $z \to x \le y \to x$. If $x \le y$ then $y \to x = 1$, hence $z \to x \le 1$, and so $x \le z$, therefore (2) holds.

Proposition

For a Hilbert Algebras in BCK-algebra $(H, \to, 1)$, we have

$z \to (y \to x) = y \to (z \to x)$.

Proof

By BCI-$2'$ it has $(z \to x) \to x \le z$, Making use of

Proposition

(1) and BCI-$1'$

$z \to (y \to x) \le ((z \to x) \to x)$

$\to (y \to x) \le y \to (z \to x)$

Since x, y, z are arbitrary, interchanging y, z in the above inequality it obtains

$z \to (y \to x) \ge y \to (z \to x)$

By BCI-$4'$ it has $z \to (y \to x) = y \to (z \to x)$, The proof is complete.

Proposition

For a Hilbert Algebras in BCK-algebra $(H, \to, 1)$, for any x, y, z in H, the following hold:

(1) $y \to x \le z$ implies $z \to x \le y$

(2) $(z \to y) \to (z \to x) \le (y \to x)$

(3) $x \le y$ implies $(z \to x) \le (z \to y)$

(4) $y \to x \le x$

(5) $1 \to x = x$

Proof

(1) is directly consequence of Proposition1.2.

(2) follows from BCI-$1'$ and (1) Let $x \le y$. Then $y \to x = 1$, and hence by (2) $(z \to y) \to (z \to x) \le y \to x \le 1$. Hence $z \to x \le z \to y$, (3) hold.

By virtue of (1), BCI-3 and BCK-5, it has $x \to (y \to x) = y \to (x \to x) = y \to 1 = 1$

Consequently $y \to x \le x$, proving (4). By BCI-$2'$, it has $(1 \to x) \to x \le 1$,

that is $x \le 1 \to x$. Moreover by (4), it gets $1 \to x \le x$, hence by BCI-$4'$, (5) hold.

For any $x, y \in H$, we denote $x \wedge y = (x \to y) \to y$, $x \wedge y$ is a lower bound of x and y,

and $x \wedge x = x$, $x \wedge 1 = 1 \wedge x = 1$, but in general $x \wedge y \ne y \wedge x$.

Proposition

In any Hilbert Algebras in BCK-algebras, we have

$(y \wedge x) \to x = y \to x$.

Proof

Since $y \wedge x \le y$, by Proposition1.2(1) we get

$y \to x \le (y \wedge x) \to x$

On the other hand, by BCI-$2'$ we have

$(y \wedge x) \to x = ((y \to x) \to x) \to x \le y \to x$

This means $(y \wedge x) \to x = y \to x$.

SUB-ALGEBRAS

Definition

Let $(H, \to, 1)$ be a Hilbert Algebras in BCK-algebras, and let H_0 be a nonempty subset of H .Then is called to be a Sub-algebras of H if for any $x, y \in H_0$, $y \to x \in H_0$.

Theorem

Suppose $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras, and let H_0 be a Sub-algebras of H, then the following hold:

(1) $1 \in H_0$,

(2) $(H_0, \to, 1)$ is also a Hilbert Algebras in BCK-algebras,

(3) H is a Sub-algebras of H,

(4) $\{1\}$ is a Sub-algebras of H.

Theorem

Given a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, let $h_0 \neq 1$, then $(\{1, h_0\}; \to 1)$ is a Sub-algebras of H.

Lemma

In a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, for any $x, y, z \in H$, the following conditions hold:

(1) if $x \neq y$ then $x \to y \neq 1$ whenever $y \to x = 1$,

(2) $y \to x = z$ implies $x \to z = 1$.

Proof

Suppose $x \neq y$ and $y \to x = 1$. If $x \to y = 1$, by BCI-4 we obtain $x = y$, which contradicts to. Hence (1) holds.

If $y \to x = z$, it follows from proposition 1.4(4) that $x \to z = x \to (y \to x) = 1$.

This shows that(2) is true. The proof is completed.

Definition

For a $n-$sequence $a_1, a_2, \cdots, a_n$ of a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, the $(n-1) \times n$ matrix

$$\mathbf{H} = \begin{pmatrix} a_2 \to a_1 & a_1 \to a_2 & \cdots & a_1 \to a_n \\ a_3 \to a_1 & a_3 \to a_2 & \cdots & a_3 \to a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n \to a_1 & a_n \to a_2 & \cdots & a_{n-1} \to a_n \end{pmatrix}$$

is called the adjoint matrix relative to the $n-$sequence $a_1, a_2, \cdots, a_n$.

Theorem

Given $n-$sequence $a_1, a_2, \cdots, a_n$ $(n \geq 2)$ for a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, if $a_i \neq a_j$ whenever $i \neq j (1 \leq i, j \leq n)$ then there exist at least a column in the adjoint matrix H which consists of non-one element.

Proof

If $n = 2$, then

$$\mathbf{H} = (a_2 \to a_1, a_1 \to a_2)$$

Suppose $a_2 \to a_1 = a_1 \to a_2 = 1$, then we have $a_1 = a_2$. Hence the assertion holds for $n = 2$.

Suppose that the assertion holds for $n = k$. For a $k+1$-sequence $a_1, a_2, \cdots, a_k, a_{k+1}$ of H,

Let $a_i \neq a_j$ whenever $i \neq j (1 \leq i, j \leq k+1)$ and its adjoint matrix

$$\mathbf{H}_{k+1} = \begin{pmatrix} a_2 \to a_1 & a_1 \to a_2 & \cdots & a_1 \to a_k & a_1 \to a_{k+1} \\ a_3 \to a_1 & a_3 \to a_2 & \cdots & a_2 \to a_k & a_2 \to a_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_k \to a_1 & a_k \to a_2 & \cdots & a_{k-1} \to a_k & a_{k-1} \to a_{k+1} \\ a_{k+1} \to a_1 & a_{k+1} \to a_2 & \cdots & a_{k+1} \to a_k & a_k \to a_{k+1} \end{pmatrix}$$

Denote

$$\mathbf{H}_k = \begin{pmatrix} a_2 \to a_1 & a_1 \to a_2 & \cdots & a_1 \to a_k \\ a_3 \to a_1 & a_3 \to a_2 & \cdots & a_3 \to a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_k \to a_1 & a_k \to a_2 & \cdots & a_{k-1} \to a_k \end{pmatrix}$$

Obviously H_k is the adjoint matrix of the k-sequence $a_1, a_2, \cdots, a_k$. By the hypothesis of induction we know that there is at least a column in H_k, which consists of non-one elements, without loss of any generality suppose that it is the first column, thus

$$\begin{cases} a_2 \to a_1 \neq 1 \\ a_3 \to a_1 \neq 1 \\ \quad \vdots \\ a_k \to a_1 \neq 1 \end{cases}$$

If $a_{k+1} \to a_1 \neq 1$ then each element in the first column of H_{k+1} is not equal to 1, and consequently the assertion holds for $n = k+1$.

If $a_{k+1} \to a_1 = 1$, then by Lemma 2.4 we obtain $a_1 \to a_{k+1} \neq 1$ as $a_1 \neq a_{k+1}$.

We can verify $a_i \to a_{k+1} \neq 1$ for $2 \leq i \leq k$.

In fact, if there is $i_1 (2 \leq i_1 \leq k)$ such that $a_{i_1} \to a_{k+1} = 1$, by Proposition1.2(2) we get $a_{i_1} \to a_1 = 1$ as $a_{k+1} \to a_1 = 1$. This is impossible. This shows that each element in the first column of H_{k+1} is not equal to 1, and so the assertion holds for $n = k+1$. This finishes the proof.

For a set H denote the cardinal of H by $|H|$. For a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, $|H|$ is called to be the order of this algebra. If $|H| < \infty$, then $(H, \to, 1)$ is called to be of finite order; if $|H| = n$, then it is said to be of order n; if $|H| = \infty$, then it is said to be of infinite order.

Theorem

Any Hilbert Algebras in BCK-algebras $(H, \to, 1)$ with order $n+1$ must contain a sub-algebra with order $n (n \geq 1)$.

Proof

Suppose $H = \{1, a_1, a_2, \cdots, a_n\}$ is a Hilbert Algebras in BCK-algebras $(H, \to, 1)$, with order $n+1$, in which $a_i \neq a_j$ whenever $i \neq j$, and let

$$\mathbf{H} = \begin{pmatrix} \mathbf{a_2 \to a_1} & \mathbf{a_1 \to a_2} & \cdots & \mathbf{a_1 \to a_n} \\ \mathbf{a_3 \to a_1} & \mathbf{a_3 \to a_2} & \cdots & \mathbf{a_3 \to a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a_n \to a_1} & \mathbf{a_n \to a_2} & \cdots & \mathbf{a_{n-1} \to a_n} \end{pmatrix}$$

be the adjoint matrix of the n-sequence $a_1, a_2, \cdots, a_n$. By Theorem 2.6 there exists at least a column in H which consists of non one elements, without loss of generality, we can suppose it is the n-th column of H, that is,

$\mathbf{a_i \to a_n \neq 1 \quad i = 1, 2, \cdots, n-1}$.

We now show that $H_0 = \{1, a_1, a_2, \cdots, a_{n-1}\}$ is a sub-algebra of H. In fact, if not, then there are i and $j (1 \leq i, j \leq n-1)$ such that $i \neq j$ and $a_j \to a_i = a_n$. By Lemma 2.4(1) we have

$\mathbf{a_i \to a_n = 1 \quad i = 1, 2, \cdots, n-1}$.

Which contradicts to $a_i \to a_n \neq 1 (i = 1, 2, \cdots, n-1)$. This completes the proof.

Theorem

Let H be a Hilbert Algebras in BCK-algebras $(H, \to, 1)$ with order $n (\geq 1)$. Then we have

$\mathbf{1 \leq N(i) \leq C_{n-1}^{i-1}, i = 1, 2, \cdots, n}$,

Where $N(i)$ denotes the number of sub-algebras with order i in H.

Proof

We know that any sub-algebra with order $i (1 \leq i \leq n)$ consists of 1 and $i-1$ non one elements, it follows that $N(i) \leq C_{n-1}^{i-1}$. On the other hand, by Theorem 2.7 H at least contains a sub-algebras H_{n-1} with order $n-1$, H_{n-1} contains a sub-algebras H_{n-2} with order $n-2$, repeating this argument we get that $1 \leq N(i)$ for all $i (1 \leq i \leq n)$. This finishes the proof.

Example

Let $H = \{1, a, b, c\}$ in which $\to$ is given by the table:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

Then $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras, the lattice of sub-algebras in H is as follows:

It is not difficult to see that

$\mathbf{N(i) = C_{4-1}^{i-1}, i = 1, 2, 3, 4}$.

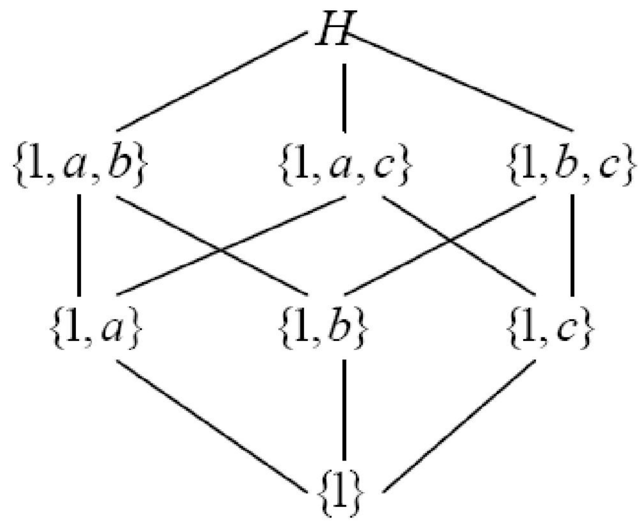
This example show that in

Theorem

$\mathbf{N(i) = C_{n-1}^{i-1}}$ may hold.

Example

Let $H = \{1, 2\}$ in which $\to$ is given by:



$1 \to 1 = 2 \to 1 = 2 \to 2 = 1$ and $1 \to 2 = 2$.

Then $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras, and $1 = N(i) = C_{2-1}^{i-1}, i = 1,2$.

This example show that in

Theorem

N(i) = 1 may hold.

Example

Let $H = \{1, a, b, c, d\}$ in which $\to$ is given by the table:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	b
c	1	1	1	1	b
d	1	a	1	c	1

Then $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras, and $1 < N(3) = C_{5-1}^{3-1}$.

This example show that in Theorem 2.8, $N(i) > 1$ and $N(i) < C_{n-1}^{i-1}$.

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