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STUDY OF CONTROLLABILITY OF MATRIX INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES

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ABSTRACT

In this paper, first we develop the method of variation of parameters formula for integro-differential system on time scales in terms of resolvent kernals and then offer sufficient conditions for the complete controllability of the first order matrix integro-differential system on time scales. 2000 Mathematics subject classification: 49K15, 93B05 and 37N35,

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INTRODUCTION

The importance of matrix dynamical systems on time scales is an interesting area of current research, which arise in number of areas of control engineering problems, dynamical systems, and feedback systems are well known. There are many results from differential equations that carry over quite naturally and easily to difference equations, while others have a completely different structure for their continuous counterparts. The study of dynamical systems on time scales sheds new light on the discrepancies between continuous and discrete dynamical systems. It is also prevents one from proving a result twice, once for continuous and once for discrete systems. The general idea, which is the main goal of Bhoner and Peterson's introductory text[1] is to prove a result for a first order differential equation when the domain of the unknown function is so-called timescale. The main objective of this paper is to explore the techniques controllability of matrix integro-differential equations on time scales.

In this paper, we focus our attention to study the controllability of the first order matrix dynamic integro-differential equation on time scales of the form.

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$$Z^{\Delta}(t) = A(t)\, Z(t) + Z(\sigma(t))\, B(t) + \int_{t_0}^{t} [K_1\,(t,s)\, Z(s) + Z(\sigma(s))\, K_2(t,s)]\, \Delta s + C(t)\, U(t)\, D(t)$$

Satisfying $Z(t_0) = Z_0$ …(1.1)

Where Z(t) is an n × n matrix, U(t) is an m × n input piecewise rd-continuous matrix called control and K(t) is an r × n output matrix. Here A(t), B(t), C(t) and D(t) are n × n, n × n and n × n and n × n matrices, respectively. Z^{Δ} (t) is the generalized delta derivative of Z, t is from a time scale T, which is a known non-empty closed subset of R. where K_1 (t,s) and K_2 (t,s) are rd-continuous square matrices of the order (n × n) and $(t,s) \in R_+^2$ and $F \in C_{rd}\,[T^+, R^{n\times n}]$.

This paper is well organized as follows: In section 2 we study some basic properties of time scale calculus and then develop the variation of parameters formula for integro-differential system.

$$Z^{\Delta}(t) = A(t)Z(t) + Z(\sigma(t))B(t) + \int_{t_0}^{t} [K_1(t,s)Z(s) + Z(\sigma(s))K_2(t,s)]\Delta s + F(t) \qquad \ldots(1.2)$$

satisfying Z $(t_0) = Z_0$

In section 3 we present sufficient conditions for the controllability of the first order matrix integro-differential system on time scales (1.1). Controllability of dynamical systems of the type (1.1) with B(t) = O (null matrix) were studied Jhon davis[3]. The results obtained in this paper include some of the results of[3,7] with B(t) = 0 and Z is a vector in (1.1).

Section 2

In this section, we give a short over view on some basic results on the time scales and we develop the method of variation of parameters formula for integro-differential system on time scales in terms of resolvent kernals and then offer sufficient conditions for the controllability of the first order matrix integro-differential system on time scales.

Now we introduce some basic definitions and results on time scales[1,2] needed in our subsequent discussion.

A timescale T is a closed subset of R; and examples of time scales include N; Z; R, Fuzzy sets etc. The set Q = $\{t \in R/Q, 0 \le t \le 1\}$ are not time scales. Time scales need not necessarily be connected. In order to overcome this deficiency, we introduce the notion of jump operators. Forward (backward) jump operator σ(t) of t for t < sup T, (respectively ρ(t) at t for t > inf T) is given by $\sigma(t) = \inf\{s \in T : s > t\}$, $(\rho(t) = \sup\{s \in T : s < t\})$, for all $t \in T$. The graininess function $\mu : T \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. Throughout we assume that T has a topology that it inherits from the standard topology on the real number R. The jump operators σ and ρ allow the classification of points in a time scale in the way: If $\sigma(t) > t$, then the point t is called right scattered ; while if $\rho(t) < t$, then t is termed left scattered. If $t < \sup T$ and $\sigma(t) = t$, then the point ' t' is called right dense: while if $t > \inf T$ and $\rho(t) = t$, then we say 't' is left-dense. We say that $f : T \to R$ is rd-continuous provided f is continuous at each right-dense point of T and has a finite left-sided limit at each left-dense point of T and will be denoted by Crd.

A function f: $T \to T$ is said to be differentiable at $t \in T^k = \{T\backslash(\rho(t)\max(T), \max t)\}$ if $\lim_{\sigma(t)\to s} \frac{f((\sigma(t) - f(s))}{\sigma(t) - s}$ where $s \in T\text{-}\{\sigma(t)\}$ exist and is said to be differentiable on T provided it is differentiable for each $t \in T^k$. A function $F : T \to T$, with $F^{\Delta}(t) = f(t)$ for all $t \in T^k$ is said to be integrable, if $\int_{t_0}^{t} f(\tau)\,\Delta\tau = F(t) - F(s)$ where F is anti derivative of f and for all s, $t \in T$. Let f: $T \to T$, and if T = R and a, $b \in T$, then $f^{\Delta}(t) = f'(t)$ and $\int_a^b f(t)\,dt = \int_a^b f(t)\,\Delta t$.

If T = Z, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ and $\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ \sum_{k=b}^{a-1} f(k) & \text{if} \quad a > b \end{cases}$

If f, g: $T \to X$ (X is a Banach space) be differentiable in $t \in T^k$. Then for any two scalars α, β the mapping $\alpha f + \beta g$ is differentiable in t and further we have:

$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t)$

$(fg)^{\Delta}(t) = (f)^{\Delta}(t)g(t) + f(\sigma(t))\,\beta\, g^{\Delta}(t)$

$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$

$(kf)^{\Delta}(t) = k\, f^{\Delta}(t)$, for any scalar k.

If f is Δ-differentiable, then f is continuous. Also if t is right scattered and f is continuous at t then –

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

We consider the matrix linear integro-differential system on time scales

$$Z^{\Delta}(t) = A(t)\, Z(t) + \int_{t_0}^{t} K_1\,(t,s)\, Z(s)\, \Delta s + F(t),\;\; Z(t_0) = Z_0 \qquad \ldots(2.1)$$

Where A(t), K_1(t,s,) are (n × n) rd-continuous matrices for $t \in T^+$ and $(t, s) \in R_+^2$ and $F \in c_{rd}\,[T^+, R^{n \times n}]$ and Z(t) is a square matrix of order n.

We get for $t_0 < s < t < \infty$

$$\varphi\,(t,s) = A(t) + \int_{s}^{t} K_1\,(t,\tau)\, \Delta\tau \qquad \ldots(2.2)$$

$$R_1\,(t,s) = I + \int_{s}^{t} R_1\,(t,\xi)\, \varphi\,(\xi\xi s)\, \Delta\xi \qquad \ldots(2.3)$$

Where I is the identity matrix order n and $K_1(t,s) = \phi\,(t,s) = R_1\,(t,s) = 0$ for $t_0 \le t < s$.

Theorem 2.1: Assume that A(t), K_1 (t,s) are (n × n) rd-continuous matrices for $t \in T^+$, (t, s) $\in T^k$ and $F \in C_{rd}\,[T^+, R^{n \times n}]$. Then the solution of (2.1) is given by[6] –

$$Z(t) = R_1\,(t,t_0)\, Z_0 + \int_{t_0}^{t} R_1\,(t,s)\, F(s)\, \Delta s \qquad \ldots(2.4)$$

with initial condition $Z(t_0) = Z_0$, where R_1 (t, s) is the unique solution of the partial differential equation on time scales

$$\frac{\partial R_1}{\Delta s}(t,s) + R_1\,(t,\sigma(s))\, A(s) + \int_{s}^{t} R_1\,(t,\sigma\,(\xi\xi))\,{}_1\,(\xi\xi s)\, \Delta\xi = 0$$

with $$R_1(t,t) = I \qquad \ldots(2.5)$$

Proof. Since ϕ is rd-continuous, it follows that R_1 in (2.4) exists and subsequently $\frac{\partial R_1}{\Delta s}$ exists and satisfies (2.5). Let Z(t) be a solution of (2.1) for $t \ge t_0$ then if we set P(s) = R_1 (t,s) Z(s), we have $P^{\Delta}(s) = \frac{\partial R_1}{\Delta s} Z(s) + R_1(t, \sigma(s)) Z^{\Delta}(s)$

$$= \frac{\partial R_1}{\Delta s} Z(s) + R_1(t,\sigma(s)) [A(s) Z(s) + \int_{t_0}^{s} K_1(s,u) Z(u) \Delta u + F(s)$$

integrating between the limits t_0 to t

$$P(t) - P(t_0) = \int_{t_0}^{t} [\frac{\partial R_1}{\Delta s} Z(s) + R_1(t,\sigma(s)) A(s) Z(s) + R_1(t,\sigma(s)) F(s)] \Delta s$$
$$+ \int_{t_0}^{t} R_1(t,\sigma(s)) [\int_{t_0}^{s} K_1(s,u) Z(u) \Delta u] \Delta s$$

$$P(t) - R_1(t_0, t) Z_0 = \int_{t_0}^{t} [\frac{\partial R_1}{\Delta s} + R_1(t, \sigma(s)) A(s) + \int_{s}^{t} R_1(t, \sigma(s)) K_1(s, u) \Delta u] Z(s) \Delta s$$
$$+ \int_{t_0}^{t} R_1(t, \sigma(s)) F(s) \Delta s$$

using (2.5), we get $P(t) = R_1(t,t_0) Z_0 + \int_{t_0}^{t} R_1(t,\sigma(s)) F(s) \Delta s$

Since P(t) = R_1 (t,t) Z(t) and R_1 (t,t) = I and

$$Z(t) = R_1(t,t_0) Z_0 + \int_{t_0}^{t} R_1(t,\sigma(s) F(s) \Delta s$$

Now, to prove that Z(t) is a solution of (2.1). Let Z(t) be the solution of (2.5) satisfying Z (t_0) = Z_0 existing for $t_0 \leq \infty$. Then

$$\int_{t_0}^{t} R_1(t,\sigma(s)) Z^{\Delta}(s) \Delta s = R_1(t,t) Z(t) - R_1(t,t_0) Z_0 - \int_{t_0}^{t} \frac{\partial R_1(t,s)}{\Delta s} Z(s) \Delta s$$

$$= \int_{t_0}^{t} R_1(t,\sigma(s)) F(s) \Delta s - \int_{t_0}^{t} \frac{\partial R_1}{\Delta s} Z(s) \Delta s$$

Using (2.5) we get –

$$\int_{t_0}^{t} R_1(t,\sigma(s))\,[Z^{\Delta}(s) - A(s)\,Z(s) - \int_{t_0}^{s} K_1(s,\xi)\,Z(\xi Z\Delta\xi - F(s)]\,\Delta s = 0$$

Since R_1 (t, σ(s)) is non-zero rd-continuous for $t_0 \leq s \leq t < \infty$. Then

$$Z^{\Delta}(s) - A(s)\,Z(s) - \int_{t_0}^{s} K_1(s,\xi)\,Z(\xi Z(\xi - F(s) = 0$$

Therefore Z(s) is a solution of $Z^{\Delta}(s) = A(s)\,Z(s) + \int_{t_0}^{s} K_1(s,\xi)\,F(\xi(\xi)$

Definition 2.1 The first order matrix dynamical systems given by (1.1) and is said to be completely controllable on [t_0, t_f], if for any given initial state $Z(t_0) = Z_0$, there exists a input signal U(t), such that the corresponding solution of (1.1) satisfies $Z(t_f) = Z_f$.

If time scale dynamical system (1.1) is controllable for all Z_0 at $t = t_0$ and for all Z_f at $t = t_f$, then the system (1.1) is said to be completely controllable.[5]

Theorem 2.2: The first order matrix integro-differential system on time scales[4]

$$Z^{\Delta}(t) = A(t)\,Z(t) + \int_{t_0}^{t} K_1(t,s)\,Z(s)\,\Delta s + C(t)\,U(t), Z(t_0) = Z_0 \qquad \ldots(2.6)$$

is completely controllable, if and only if the controllability matrix

$$\varphi(t_0,t_1) = \int_{t_0}^{t_1} R_1(t_0,\sigma(s))\,C^{*}(s)\,C(s)\,R_1^{*}(t_0\,\sigma(s))\,\Delta s$$

is a non-singular, where R_1 (t, σ(s)) is the resolvent matrix. The control function

$$U(t) = -C(t)\,R_1^{*}(t_0,t)\,\boldsymbol{\phi}^{-1}(t_0,t_1)\,[Z_0 - R(t_0,t_1)\,Z_1]$$

defined for $t_0 < t < t_1$ transfers $Z(t_0) = Z_0$ to $Z(t_1) = Z_1$

Proof: Any solution Z(t) of (2.6) is given by –

$$Z(t) = R_1(t_1,t_0)\,Z_1 + \int_{t_0}^{t} R_1(t,\sigma(s))\,C^{*}(s)\,U(s)\,\Delta s$$

and hence $$Z(t_1) = R_1(t_1, t_0)\,[Z_0 + \int_{t_0}^{t_1} R_1(t_0, \sigma(s))\, C^*(s)\, U(s)\, \Delta s]$$

$$= R_1(t_1, t_0)\,[Z_0 + \int_{t_0}^{t_1} R(t_0, \sigma(s))\, C(s)\,(-C^*(s)\, R^*(t_0, \sigma(s))\, \varphi^{-1}(t_0, t_1))\,(Z_0 - R_1(t_0 t_1)\, Z_1)\, \Delta s]$$

$$= R_1(t_1, t_0)\,[Z_0 - \varphi(t_0, t_1)\, \varphi^{-1}(t_0, t_1)\,(Z_0 - R(t_0, t_1)\, Z_1)] = Z_1$$

Similarly, we obtain the converse as in previous theorem.

Theorem 2.3: Assume that B(t) and $K_2(t, s)$ are rd-continuous $(n \times n)$ matrices for $t \in T^+$, $(t, s) \in T$ and $F \in C[R_+, R^{n \times n}]$ and then the solution of –

$$Z^{\Delta}(t) = Z(\sigma(t))\, B(t) + \int_{t_0}^{t} Z(\sigma(s))\, K_2(t,s)\, \Delta s + D(t)\, U(t),\ Z(t_0) = Z_0 \qquad \text{…(2.7)}$$

is given by $Z(t) = Z_0 R_2^*(t, t_0) + \int_{t_0}^{t} D(s)\, U^*(s)\, R_2^*(t,s)\, \Delta s$

where $R_2(t, s)$ is the resolvent kernel and is the unique solution of –

$$\frac{\partial}{\Delta s}(R_2^*(t,s)) + B(s) R_2^*(t,s) + \int_s^t K_2(\xi, s) R_2^*(t, \sigma(\xi)) \Delta\xi = 0$$

with $$R_2(t, t) = I \qquad \text{…(2.8)}$$

Theorem 2.4. The matrix integro-differential system (2.7) is completely controllable, if and only if, the controllable matrix

$$\xi(t_o, t_1) = \int_{t_0}^{t_1} R_2(t_0, \sigma(s))\, D^*(s)\, D(s)\, R_2^*(t_0\, \sigma(s))\, \Delta s$$

is non-singular, where $R_2(t, s)$ is the resolvent matrix. The control function U(t) given by –

$$U(t) = -D(t) R_2^*(t_0, t) \xi^{-1}(t_0, t)[Z_0^* - R_2(t_0, t_1) Z_1^*]$$

Defined for $t_0 < t < t_1$ transfers $Z(t_0) = Z^*{}_0$ to $Z(t_1) = Z^*{}_1$.

Section 3

We shall now consider the superposition of the above systems and present a set of sufficient conditions for the complete controllability of matrix integro-differential system (1.1) on time scales.

Theorem 3.1.[8] The matrix integro-differential system (1.1) satisfies $Z(t_0) = Z_0$ has a unique solution given by –

$$Z^{\Delta}(t) = R_1(t,t_0)\,Z_0\,R_2^*(t,t_0) + \int_{t_0}^{t} R_1(t,\boldsymbol{\sigma}(s))\,C(s)\,U(s)\,D^*(s)\,R_2^*(t,\boldsymbol{\sigma}(s))\,\Delta s \qquad \ldots(3.1)$$

where R_1 and R_2 are the solutions of partial differential equations on time scales (2.8).

Theorem 3.2. The matrix integro-differential system (1.1) is completely controllable if and only if the matrices

$$\boldsymbol{\phi}(t_0, t_1) = \int_{t_0}^{t} R_1(t_0, \sigma(s))\,C^*(s)\,C(s)\,R_1^*(t_0, \sigma(s))\,\Delta s$$

and $$\boldsymbol{\xi}(t_0, t_1) = \int_{t_0}^{t} R_2^*(t_0, \boldsymbol{\sigma}(s))\,D(s)\,D^*(s)\,R_2(t_0, \boldsymbol{\sigma}(s))\,\Delta s$$

are non-singular, where R_1 (t, s) and R_2 (t, s) resolvent matrices. The control function U(t) given by –

$$U(t) = -C(t)\,R_1^*(t_0, t)\,\varphi^{-1}(t_0, t_1)[Z_0 - R_1(t_0, t_1)\,Z_1\,R_2^*(t_1, t_0)]\,\xi^{-1}(t_0, t_1)\,R_2^*(t_0, t_1)\,D(t)$$

is defined for $t_0 < t < t_1$ transfers $Z(t_0) = Z_0$ to $Z(t_1) = Z_1$.

Proof: Any solution Z(t) of the matrix integro-differential system on time scales is given by (3.1) substituting the general form of the control U(t) in (3.1), we get –

$$Z(t_1) = R_1(t_1, t_0)\,Z_0 R_2^*(t_1, t_0) + R_1(t_1, t_0)\int_{t_0}^{t_1} R_1(t_0, \sigma(s))\,C(s)[-C^*(s)\,R_1^*(t_0, \sigma(s))\,\boldsymbol{\phi}^{-1}(t_0, t_1)$$
$$[Z_0 - R_1(t_0, t_1)\,Z_1 R_2(t_1, t_0)]\,\xi^{-1}(t_0, t_1)\,R_2^*(t, t_1)\,D(s)]\,D^*(s)\,R_2^*(t_0, \sigma(s))\,\Delta s$$

$$= R_1(t_1,t_0)Z_0R_2^*(t,t_0) - R_1(t_1,t_0)\phi(t_0,t_1)\phi^{-1}(t_0,t_1)[X_0 - R_1(t_0,t_1)Z_1R_2(t_1,t_0)] \times \xi(t_0,t_1)\xi^{-1}(t_0,t_1)R_2^*(t_0.t_1)$$

$$= R_1(t_1,t_0)Z_0R_2^*(t_1,t_0) - R_1(t_1,t_0)Z_0R_2^*(t_0,t_1) + R_1(t_1,t_0)R(t_0,t_1)Z_1R_2^*(t_1,t_0)R_2^*(t_0.t_1)$$

$$= R_1(t_1,t_1)Z_1R_2^*(t_1,t_1) = Z_1$$

Conversely suppose that the system (1.1) is completely controllable. Then it can easily be proved as in theorem 3.1[8] that the controllable matrices $\phi(t_0, t_1)$ and $\psi(t_0, t_1)$ are positive definite.

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