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Regular Ternary Semigroups

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Abstract

Intriguing properties of regular ternary semigroups and completely regular ternary semigroups were discussed in the article.

Keywords: Regular ternary semigroup; Completely regular ternary semigroup

Introduction

Los [1] concentrated a few properties of ternary semigroups and demonstrated that each ternary semigroup can be installed in a semigroup. Sioson [2] concentrated ideal theory in ternary semigroups. He likewise presented the thought of regular ternary semigroups and characterized them by utilizing the thought of quasi ideals. Santiago [3] built up the theory of ternary semigroups and semiheaps. Dutta and Kar [4,5] presented and concentrated the thought of regular ternary semirings. Jayalalitha et al. [6] presented and learned about the filters in ternary semigroups. As of late, various mathematicians have taken a shot at ternary structures. In this paper, we concentrate some intriguing properties of regular ternary semigroups and completely regular ternary semigroups.

Definition 1

An element x in a ternary semigroup T is said to be a regular if $\exists$ an element $a \in T \ni xax=x$ [2].

A ternary semigroup is said to be regular if all of its elements are regular.

Theorem 1

The following conditions in a ternary semigroup T are equivalent:

(i) T is regular.

(ii) For any right ideal R, lateral ideal M and left ideal L of T, $RML = R \cap M \cap L$.

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(iii) For $x, y, z \in T$, $\langle x\rangle_r \langle y\rangle_m \langle z\rangle_l = \langle x\rangle_r \cap \langle y\rangle_m \cap \langle z\rangle_l$.

(iv) For $x \in T$, $\langle x\rangle_r \langle x\rangle_m \langle x\rangle_l = \langle x\rangle_r \cap \langle x\rangle_m \cap \langle x\rangle_l$.

Proof

$(\text{i}) \Rightarrow (\text{ii})$ Suppose T is a regular ternary semigroup. Let R, M and L be a right ideal, a lateral ideal and a left ideal of T. Then clearly, $\text{RML} \subseteq \text{R} \cap \text{M} \cap \text{L}$. Now for $x \in \text{R} \cap \text{M} \cap \text{L}$, we have *x=xax* for some $a \in T$. This implies that $x = xax = (xax)(axa)(xax) \in RML$.

Thus, we have $\text{R} \cap \text{M} \cap \text{L} \subseteq \text{RML}$. So we find that $\text{RML} = \text{R} \cap \text{M} \cap \text{L}$.

Clearly, $(\text{ii}) \Rightarrow (\text{iii})$ and $(\text{iii}) \Rightarrow (\text{iv})$.

It remains to show that $(\text{iv}) \Rightarrow (\text{i})$.

Let $x \in T$. Clearly, $x \in \langle x\rangle_r \cap \langle x\rangle_m \cap \langle x\rangle_l = \langle x\rangle_r \langle x\rangle_m \langle x\rangle_l$.

Then we have, $x \in (xTT \cup nx)(TxT \cup TTxTT \cup nx)(TTx \cup nx) \subseteq xTx$.

So we find that $x \in xTa$ and hence there exists an elements $a \in T$ such that *x=xax*. This implies that *x* is regular and hence T is regular.

We note that every left and right ideal of a regular ternary semigroup may not be a regular ternary semigroup.
However, for a lateral ideal of a regular ternary semigroup, we have the following result:

Lemma

Every lateral ideal of a regular ternary semigroup T is a regular ternary semigroup.

Proof

Let L be a lateral ideal of regular ternary semigroup T. Then for each $x \in L$ there exists $a \in T$ such that *x=xax*. Now *x=xax=xaxax=x(axa)x=xpx* where $p = axa \in L$. This implies that L is a regular ternary semigroup.

Definition 2

An ideal A of a ternary semigroup T is said to be a regular ideal if $\text{A} \cup \text{RML} = \text{R} \cap \text{M} \cap \text{L}$ for any right ideal $R \supseteq A$, lateral ideal $M \supseteq A$ and left ideal $L \supseteq A$.

Remark 1

From Definition 2, it follows that T is always a regular ideal and any ideal that contains a regular ideal is also a regular ideal. Now if for any right ideal R, lateral ideal M and left ideal L; RML contains a regular ideal, then $\text{RML} = \text{R} \cap \text{M} \cap \text{L}$

Proposition

A ternary semigroup T is a regular ternary semigroup if and only if $\{0\}$ is a regular ideal of T.

Proof

Let P be the nuclear ideal of a ternary semigroup T. i.e., the intersection of all non-zero ideals of T, P_r is the intersection of all non-zero right ideals of T, P_m is the intersection of all non-zero lateral ideals of T and P_l is the intersection of all non-zero left ideals of T. Now if P= $\{0\}$, then clearly $P=P_r=P_m=P_l$.

Theorem 2

Let T be a ternary semigroup and $P=P_r=P_m=P_l$. Then T is a regular ternary semigroup if and only if P is a regular ideal of T.

Proof

If $P=P_r=P_m=P_l=\{0\}$, then proof follows from proposition. So we suppose that,

$P=P_r=P_m=P_l \neq \{0\}$. Let T be a regular ternary semigroup. Then from proposition, it follows that {0} is a regular ideal of T. Now, $\{0\} \subseteq P = P_r = P_m = P_l$ implies that P is a regular ideal of T, by using Remark 1.

Conversely, let P be a regular ideal of T. Then $P \cup RML = R \cap M \cap L$ for any right ideal $R \supseteq P$, lateral ideal $M \supseteq P$ and left ideal $L \supseteq P$ of T. Since PPP is a right ideal of T and $P=P_r$, we have $P = P_r \subseteq PPP \subseteq RML$.

Consequently, $P \cup RML = RML$. So $RML = R \cap M \cap L$ and hence from Theorem 2, it follows that T is a regular ternary semigroup.

Corollary 1

Let T be a ternary semigroup and $P=P_r=P_m=P_l$. Then T is a regular ternary semigroup if and only if every ideal of T is regular.

Proof

Suppose T is a regular ternary semigroup. Then from Theorem 2, it follows that P is a regular ideal of T. Now $P=P_r=P_m=P_l$ implies that every non-zero ideal of T contains the regular ideal P of T. Consequently, by using Remark 1, we find that every ideal of T is regular.

Conversely, if every ideal of T is regular, then P is a regular ideal of T and hence from Theorem 2, it follows that T is a regular ternary semigroup.

Theorem 3

The following conditions in a ternary semigroup T are equivalent:

(i) A is a regular ideal of T.

(ii) For $x, y, z \in T$, $A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l = A \cup (\langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l)$.

(iii) For $x \in T$, $A \cup \langle x \rangle_r \langle x \rangle_m \langle x \rangle_l = A \cup (\langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l)$.

(iv) For each $x \in T \setminus A = A', x = \{a\} \cup \bigcup_{i=1}^{n} xp_i xq_i x \cup \bigcup_{i=1}^{n} xr_i s_i xu_i v_i x$ for some $a \in A$ and $p_i, q_i, r_i, s_i, u_i, v_i \in T$.

Proof

(i) $\Rightarrow$ (ii) Suppose A is a regular ideal of T. We note that for $x, y, z \in T$, $A \subseteq (A \cup \langle x \rangle_r),\ (A \cup \langle y \rangle_m),\ (A \cup \langle z \rangle_l)$.

Now $A \cup \langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l \subseteq (A \cup \langle x \rangle_r) \cap (A \cup \langle y \rangle_m) \cap (A \cup \langle z \rangle_l) = A \cup (A \cup \langle x \rangle_r)(A \cup \langle y \rangle_m)(A \cup \langle z \rangle_l)$ (since A is regular).

$$\subseteq A \cup AAA \cup A\langle y \rangle_m A \cup A\langle y \rangle_m \langle z \rangle_l \cup AA\langle z \rangle_l \cup \langle x \rangle_r AA \cup \langle x \rangle_r A\langle z \rangle_l \cup \langle x \rangle_r \langle y \rangle_m A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l$$
$$\subseteq A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l .$$

Again $\langle x \rangle_r \langle y \rangle_m \langle z \rangle_l \subseteq \langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l$ implies that $A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l \subseteq A \cup \langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l$.

So we find that $A \cup \langle x \rangle_r \langle y \rangle_m \langle z \rangle_l = A \cup (\langle x \rangle_r \cap \langle y \rangle_m \cap \langle z \rangle_l)$.

(ii) $\Rightarrow$ (iii) Put y=z=x in (ii) we get (iii).

(iii) $\Rightarrow$ (iv) We first note that $\langle A \cup \langle x \rangle_r \rangle_r = A \cup \langle x \rangle_r = A \cup \langle x \rangle_r \cap T \cap T = A \cup \langle x \rangle_r TT$

$$= A \cup (xTT \cup nx)TT = A \cup xTTTT \cup nxTT = A \cup \langle xTT \rangle_r = A \cup xTT$$

Similarly we have, $\langle A \cup \langle x \rangle_m \rangle_m = A \cup TxT \cup TTxTT$ and $\langle A \cup \langle x \rangle_l \rangle_l = A \cup TTa.$

Now $\langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l \subseteq \langle A \cup \langle x \rangle_r \rangle_r \cap \langle A \cup \langle x \rangle_m \rangle_m \cap \langle A \cup \langle x \rangle_l \rangle_l$

$$\subseteq A \cup (\langle A \cup \langle x \rangle_r \rangle_r \cap \langle A \cup \langle x \rangle_m \rangle_m \cap \langle A \cup \langle x \rangle_l \rangle_l)$$
$$= A \cup (\langle A \cup \langle x \rangle_r \rangle_r \cdot \langle A \cup \langle x \rangle_m \rangle_m \cdot \langle A \cup \langle x \rangle_l \rangle_l)$$
$$= A \cup (A \cup xTT)(A \cup TxT \cup TTxTT)(A \cup TTx)$$
$$\subseteq A \cup (xTxTx \cup xTTxTTx)$$

Since, $x \in \langle x \rangle_r \cap \langle x \rangle_m \cap \langle x \rangle_l$ there exists $a \in A$ and $p_i, q_i, r_i, s_i, u_i, v_i \in T$ such that $x = \{a\} \cup \bigcup_{i=1}^{n} xp_i xq_i x \cup \bigcup_{i=1}^{n} xr_i s_i xu_i v_i x$.

(iv) $\Rightarrow$ (i) Let R, M and L be any right, lateral and left ideal of T respectively such that $R, M, L \supseteq A$. Then clearly, $A \cup RML \subseteq R \cap M \cap L$. Again, let $x \in R \cap M \cap L$. Then by using condition (iv), we have $x = \{a\} \cup \bigcup_{i=1}^{n} xp_i xq_i x \cup \bigcup_{i=1}^{n} xr_i s_i xu_i v_i x$ for some $a \in A$ and $p_i, q_i, r_i, s_i, u_i, v_i \in T$. Since $\bigcup_{i=1}^{n} xp_i xq_i x,\ \bigcup_{i=1}^{n} xr_i s_i xu_i v_i x \in RML,\ x \in A \cup RML$ and hence $R \cap M \cap L \subseteq A \cup RML.$ Thus $A \cup RML = R \cap M \cap L$. Consequently, A is a regular ideal.

Theorem 4

Let A be a regular ideal of a ternary semigroup T. For any right ideal R, lateral ideal M and left ideal L of T, if $\mathrm{RML} \subseteq \mathrm{A}$ then $\mathrm{R} \cap \mathrm{M} \cap \mathrm{L} \subseteq \mathrm{A}$.

Proof

Suppose for any right ideal R, lateral ideal M and left ideal L of T, $\mathrm{RML} \subseteq \mathrm{A}$, where A is a regular ideal of T. Then $A \subseteq (A \cup R), (A \cup M), (A \cup L)$.

Now $\mathrm{R} \cap \mathrm{M} \cap \mathrm{L} \subseteq (A \cup R) \cap (A \cup M) \cap (A \cup L)$

$= A \cup ((A \cup R)(A \cup M)(A \cup L))$ [Since A is regular]

$\subseteq A \cup AAA \cup AAL \cup AMA \cup AML \cup RAA \cup RAL \cup RMA \cup RML$

$\subseteq A$.

From Theorem 4, we have the following results:

Corollary 2

A regular and strongly irreducible ideal of a ternary semigroup T is a prime ideal of T.

Corollary 3

Every regular ideal of a ternary semigroup T is a semi prime ideal of T.

Theorem 5

A ternary semigroup T is regular if and only if every ideal of T is idempotent.

Proof

Let T be a regular ternary semigroup and A be any ideal of T. Then $\mathrm{A}^3 = AAA \subseteq TTA \subseteq A$. Let $x \in A$. Then there exists $a \in T$ such that *x=xax=xaxax*. Since A is an ideal and $x \in A$, $axa \in A$. Thus $x = xax = xaxax \in A^3$. Consequently, $A \subseteq A^3$ and hence $\mathrm{A}^3 = AAA = A$ i.e., A is idempotent.

Conversely, suppose that every ideal of T is idempotent. Let P, Q and R be three ideals of T. Then $PQR \subseteq PTT \subseteq P$, $PQR \subseteq TQT \subseteq Q$ and $PQR \subseteq TTR \subseteq R$. This implies that $PQR \subseteq P \cap Q \cap R$. Also, $(P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R) \subseteq PQR$. Again, since $(P \cap Q \cap R)$ is an ideal of T, $(P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R) = P \cap Q \cap R$. Thus $P \cap Q \cap R \subseteq PQR$ and hence $P \cap Q \cap R = PQR$. Therefore, by Theorem 2, T is a regular ternary semigroup.

Theorem 6

A ternary semigroup T is left (resp. right) regular if and only if every left (resp. right) ideal of T is completely semiprime.

Proof

Let T be a left regular ternary semigroup and L be any left ideal of T. Suppose $a^3 = aaa \in L$ for $a \in T$. Since T is left regular, there exists an element $x \in T$ such that $a = xaa = x(xaa)a = xx(aaa) \in TTL \subseteq L.$ Thus L is completely semiprime.

Conversely, suppose that every left ideal of T is completely semiprime. Now for any $a \in T$, Taa is a left ideal of T. Then by hypothesis, Taa is a completely semiprime ideal of T. Now $a^3 = aaa \in Taa$. Since Taa is completely semiprime, it follows that $a \in Taa$. So there exists an element $x \in T$ such that a=xaa. Consequently, a is left regular. Since a is arbitrary, it follows that T is left regular.

Equivalently, we can prove the Theorem for right regularity.

Completely Regular Ternary Semigroup

Definition 3

A pair (p, q) of elements in a ternary semigroup T is known as an idempotent pair if $pq(pqx)=pqx$ and $(xpq)pq=xpq$ for all $x \in T$ [3].

Definition 4

Two idempotent pairs (p, q) and (r, s) of a ternary semigroup T are known as an equivalent, if $pqx=rsx$ and $xpq=xrs$ for all $x \in T$ [3].

In notation we write (p, q) ~ (r, s).

Definition 5

An element x of a ternary semigroup T is said to be completely regular if $\exists$ an element $a \in T \ni xax = x$ and the idempotent pairs (a, x) and (x, a) are equivalent.

If all the elements of T are completely regular, then T is called completely regular [3].

Definition 6

An element x of a ternary semigroup T is known as a left regular if $\exists$ an element $a \in T \ni axx=x$

Definition 7

An element x of a ternary semigroup T is said to be right regular if $\exists$ an element $a \in T \ni xxa = x$

Theorem 7

A ternary semigroup T is completely regular then T is left and right regular. [i.e., $x \in x^2T \cap Tx^2$ for all $x \in T$].

Proof

Suppose T is a completely regular ternary semigroup. Let $x \in T$. Then $\exists$ an element $a \in T \ni xax = x$ and the idempotent pairs (*x*, *a*) and (*a*, *x*) are equivalent i.e., *xab=axb* and *bxa=bax* for all $b \in T$. Now in particular, putting *b=x* we find that *xax=axx* and *xaa=xax*. This implies that $x \in xxT$ and $x \in Txx$. Hence T is left and right regular.

Theorem 8

A ternary semigroup T is left and right regular then $x \in x^2Tx^2$ for all $x \in T$.

Proof

Suppose that T is both left and right regular. Let $x \in T$. Then $\exists p, q \in T \ni x = xxp$ and $x=qxx$. This implies that $xpz=qxxpz=qxz$ for all $z \in T$.

Now $x=xxp=x(xxp)p=x^2(xpp)=x^2(qxxpp)=x^2(qxp)=x^2q(qxx)p=x^2 \; q^2(xxp)=x^2 \; q^2x=x^2 \; q^2qxx=x^2 \; q^3x^2 \in x^2Tx^2$. Hence $x \in x^2Tx^2$ for all $x \in T$.

Theorem 9

If T is ternary semigroup $x \in x^2Tx^2$ for all $x \in T$ then T is completely regular.

Proof

Suppose $x \in x^2Tx^2$ for all $x \in T$. Then $\exists a \in T \ni x = x^2ax^2$

Now $x = x^2ax^2 = x(xax)x = xba$, where $b = xax \in T$. This implies that T is regular. Also $xbc = x(xax)c = x^2ax^2c$ and $bxc = (xax)xc = x^2ax^2c$ for all $c \in T$. This shows that the idempotent pairs (*x*, *b*) and (*b*, *x*) are equivalent.

Consequently, T is a completely regular ternary semigroup.

Definition 8

A sub semigroup S of a ternary semigroup T is said to be a bi-ideal of T if $STSTS \subseteq S$.

Theorem 10

A ternary semigroup T is completely regular ternary semigroup if and only if every bi-ideal of T is completely semiprime.

Proof

Let T is a completely regular ternary semigroup. Let P be any bi-ideal of T. Let $p^3 \in P$ for $p \in T$. Since T is completely regular, from Theorem 10, it follows that $p \in p^2Tp^2$. This implies that there exists $x \in T$ such that $p = p^2xp^2 = p(p^2xp^2)x(p^2xp^2)p = p^3(xp^2x)p(p^2xp^2)xp^3 = p^3(xp^2x)p^3(xp^2x)p^3 \in PTPTP \subseteq P$. This shows that P is completely semiprime.

Conversely, assume that every bi-ideal of T is completely semiprime. Since every left and right ideal of a ternary semigroup T is a bi-ideal of T, it follows that every left and right ideal of T is completely semiprime. Consequently, we have from Theorem 6 that T is both left and right regular. Now by using Theorem 9, we find that T is a completely regular ternary semigroup.

Theorem 11

If T is a completely regular ternary semigroup, then every bi-ideal of T is idempotent.

Proof

Let T be a completely regular ternary semigroup and P be a bi-ideal of T. Clearly T is a completely regular ternary semigroup. Let $p \in P.$ Then there exists $x \in T$ such that *p=pxp*. This implies that $p \in PTP$ and hence $P \subseteq PTP.$ Also $PTP \subseteq PTPTP \subseteq P.$ Thus we find that *P=PTP*. Again, we have from Theorem 11 that $p \in p^2Tp^2 \subseteq P^2TP^2.$ This implies that $p \subseteq P^2TP^2 = P(PTP)P = PPP \subseteq P$. Hence $P^3 \in P.$ Therefore every bi-ideal of P is idempotent.

Conclusion

Ternary structures and their speculation, the purported *n*-ary structures bring certain expectations up in perspective of their conceivable applications in organic chemistry.

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