Int. J. Chem. Sci.: 14(4), 2016, 3005-3012
ISSN 0972-768X
www.sadgurupublications.com

OSCILLATORY AND NON OSCILLATORY PROPERTIES OF FOURTH ORDER DIFFERENCE EQUATION

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ABSTRACT

This paper establish the various properties of solution of fourth order difference equation of the form

$$\Delta^2(p_n\Delta^2 y_n) + q_{n+1}\Delta^2 y_{n+1} + r_{n+2}y_{n+2} = 0$$

Where $p_n, q_n \& r_n$ are real sequences satisfying $p_n > 0, q_n \geq 0 \& r_n > 0$ for each n ≥ 0.

Key words: Oscillation, Non-oscillation, Difference equation, Trivial and nontrivial solution, Generalized zero.

INTRODUCTION

Consider the fourth order difference equation of the

$$\Delta^2(p_n\Delta^2 y_n) + q_{n+1}\Delta^2 y_{n+1} + r_{n+2}y_{n+2} = 0 \quad \ldots(1)$$

Where $p_n, q_n \& r_n$ are real sequences satisfying $p_n > 0, q_n \geq 0 \& r_n > 0$ for each n ≥ 0 and the forward difference operator Δ is defined by $\Delta y_n = y_{n+1} - y_n$ also $y_n = y(n)$.

Definition 1: Let $y_n$be a function defined on N, we say $k \in N$ is a generalized zero for y_n if one of following holds:

(i) $y_n = 0$

(ii) $k\ N(1)\ and\ y_{n-1}y_n\ 0, k\ N(1)$, and there exists an integer m, such that $1 < m \leq k$.

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(iii) $(-1)^m y_{k-m} y_n > 0, and\ y_j = 0\ for\ all\ j \in N(k-m+1, k-1)$

A generalized zero for y_n is said to be of order 0, 1, or $m > 1$, according to whether condition (i), (ii) or (iii), respectively, holds. In particular, a generalized zero of order 0 will simply be called a zero, and a generalized zero of order one will again be called a node.

Obviously, if $y(a) = y(a+1) = y(a+2) = y(a+3) = 0$ for some $a \in N$, then $y_n \equiv 0$ is the only solution of (1). Thus, a nontrivial solution of (1) can have zeros at no more than three consecutive values of k. In Definition 1 we shall show that a nontrivial solution of (1) cannot have a generalized zero of order $m > 3$. However, a solution of (1) can have arbitrarily many consecutive nodes, as it is clear from $y_n = (-1)^n$, which is a solution of (1).

The following properties of the solutions of (1) are fundamental and will be used subsequently.

(S_1) If y_n is a nontrivial solution of (1) and if

$(a)\ y_n \geq 0$ $(b)\ \Delta y_n \geq 0$ $(c)\ \Delta^2\ y_{n+1} \geq 0$ $(d) \Delta^3 y_{n+2} \geq 0$

For some $k = a \in N(2)$, then $(a), (b), (c) \& (d)$ holds for all $k \in N(a)$, with strict inequality in (a) for all $k \in N(a+2)$, strict inequality in (b) for all $k \in N(a+1)$, and strict inequality in (c) and (d) for all $k \in N(a+3)$. Furthermore,

$$\Delta^2(p_n \Delta^2 y_n) + q_{n+1} \Delta^2 y_{n+1} + r_{n+2} y_{n+2} \geq 0\ \ for\ all\ k \in N(a) \qquad \ldots(2)$$

With strict inequality for all $k \in N(a+2)$, and y_n, Δy_n, and $\Delta^2 y_n$ all tend to $\infty\ as\ k \to \infty$.

$(S_2)\ If\ y_n\ is\ a$ Nontrivial solution of (1) and if

$(a_1)\ y_n \geq 0$ $(b_1)\ \Delta y_n \geq 0$ $(c_1)\ \Delta^2 y_n\ \geq 0$ $(d_1)\ \Delta^3 y_n \geq 0$

For some $k = a \in N$, then $(a_1), (b_1), (c_1)$ & (d_1) holds for all $k \in N(a)$, with strict inequality in $(a_1), (b_1), (d_1)$ for all $k \in N(a+3)$, and in (c_1) for all $k \in N(a+4)$. Furthermore,

$$\Delta^4 y_n \geq 0\ for\ all\ k \in N(a) \qquad \ldots(3)$$

With strictly inequality for all $k \in N(a+2)$, and $y_n, \Delta y_n, \& \Delta^2 y_n$ all tend to ∞ k→∞.

(S_3) If y_n is a nontrivial solution of (1) and if

$(a_2) y_n \geq 0$ $(b_2)\, \Delta y_{n+1} \leq 0$ $(c_2)\, \Delta^2\, y_{n+1} \geq 0$ $(d_2)\, \Delta^3 y_{n+1} \leq 0$

For some $k = a \in N(3)$, then (2) holds for all $k \in N(2, a)$, and

$$\Delta^2(p_n \Delta^2 y_n) + q_{n+1}\Delta^2 y_{n+1} + r_{n+2} y_{n+2} \geq 0 \;\; for\ all\ k \in N(2, a) \qquad ...(4)$$

Furthermore, $y(0) > y(1) > 0, and\ \Delta y(0) < 0$. Strict inequality holds in (a_2) and (3) for all $k \in N(2, a-2)\ if\ a \in N(4)$, in (b_2) for all $k \in N(2, a-1)$, and in (c_2) for all $k \in N(2, a-3)$ if $a \in N(5)$.

(S_4) Let $a \in N(2). If\ y_n$ is a solution of (1) with $y(a) = 0, y(a-1) \geq 0, y\,(a+1) \geq 0, y(a-1)$ and $y(a+1)$ not both zero, then at least one of the following conditions must be true. (i) Either $y_n > 0\ for\ all\ k \in N(a+2)$, or (ii) $y_n < 0$ for all $k \in N(0, a-1)$. In particular, y_n cannot have generalized zeros of any order at both α and β, where $\alpha \in N(0, a-1)\ and\ \beta \in N(a+2)$. An analogous statement holds for the hypotheses $y(a-1) \leq 0\ and\ y(a+1) \leq 0$.

RESULTS AND DISCUSSION

Theorem 1.1. If y_n is a nontrivial solution of (1) with zeros at three consecutive values of k, say $a, a+1 \& a+2$ then y_n has no other generalized zeros. If $y(a+3) > 0 (< 0)$, then $\Delta y_n \geq 0 (\leq 0)$ foe all k, and the inequality is strict if $k \in N(a+2)\ or\ k \in N(0, a-1)$. In particular, if

$$\alpha \in N(0, a-1) \text{ and } \beta \in N(a+3), \text{ then } y(\alpha)y(\beta) < 0.$$

Proof. Clearly $\Delta y(a) = \Delta^2 y(a) = 0$. Since the solution y_n is nontrivial, we may assume that $y(a+3) > 0$. Thus, $\Delta^3 y(a) > 0$ and by (S_2), y_n is positive and strictly increasing on $N(a+3)$. Next, let $v_n = -y_n$. Then $v(a+1) = 0, \Delta v(a) = 0, \Delta^2 v(a) = 0\ and\ \Delta^3 v(a) < 0$. If $a \in N(2)$, then (S_3) implies that v_n is positive and strictly decreasing on $N(a, 0)$. Thus y_n is negative and strictly increasing on $N(a, 0)$. If a=1, then we again assume that $y(a+3) = y(4) > 0$. Then by (1) $\Delta^4 y(0) = r(2)y(2) = 0$. But, $\Delta^4 y(0) = y(4) + y(0)$, so $y(0) = -y(4) < 0\ and\ \Delta y(0) = y(1) - y(0) > 0$, as claimed. If a=0, then the part of the conclusion concerning $k \leq a-1$ is empty. This completes the proof.

Theorem 1.2. Let $a \in N(1)$, suppose that y_n is a solution of (1) with $y(0) = 0, y(a+1) = 0, y(a+2) \neq 0, but\ a+2$ is a generalized zero for y_n. Then y_n has no other generalized zeros.

If $y(a+2) > 0(< 0)$, then $\Delta y_n \geq 0(\leq 0)$ for all $k \in N$, with strict inequality for all $k \in N(a+2) or\ k \in N(0, a-1)$. In particular, if $\alpha \in N(0, a-1)$ and $\beta \in N(a+2), then\ y(\alpha)y(\beta) < 0$.

Proof. Since $y(a+2) \neq 0$, we can assume that $y(a+2) > 0$. since $y(a) = y(a+1) = 0, a+2$ cannot be a generalized zero of order 1 or 2, and theorem (1) implies that the order cannot be greater than 3. Thus, a+2 is a generalized zero of order 3, which implies that $y(a-1) < 0$, now since from (1), we have $\Delta(p_n \Delta^2 y_n) > 0$, it follows that

$\Delta^3 y(a) > 0, clearly\ \Delta^2 y(a) > 0, \Delta y(a) = 0$ and $y(a) = 0$, thus by $(S_2), y_n$ is positive and strictly increasing on $N(a+3)$. For $k \in N(0, a), let\ v_n = -y_n$. Then $v(a) = 0, \Delta v(a-1) < 0, \Delta^2 v(a-1) > 0$ and $\Delta^3 v(a-1) < 0$. If $a \in N(3)$, then as in equation (1), (S_3) yields the results. If a=2, then $y(2) = y(3) = 0, y(1) < 0, y(4) > 0$ and $\Delta y(1) > 0$. By (1) we have $\Delta^4 y(0) = 0$. But, $\Delta^4 y(0) = y(4) - 4y(3) + 6y(2) - 4y(1) + y(0) = y(4) - 4y(1) + y(0)$, and so $4y(1) - y(0) = y(4) > 0$. Hence, $y(0) < 4y(1) < 0$, and $y(0) - y(1) < 3y(1) < 0$.

Therefore, $y(0) < 0$ and $\Delta y(0) > 0$, as claimed. If a=1, then $y(1) = y(2) = 0, y(3) \neq 0$, and $a+2 = 3$ is a generalized zero. It follows from the definition of a generalized zero that this must be a generalized zero of order 3, so that if $y(3) > 0$ then $y(0) < 0$. Hence $\Delta y(0) > 0$, which complete the proof.

Corollary 1.3. If y_n is a nontrivial solution of (1) with generalized zeros at α and β and a zero at a, where $\alpha + 1 < a < \beta - 1$, then $y(a-1)y(a+1) < 0$. In particular, y_n does not have a generalized zero at a+1.

Proof. Since $\alpha + 1 < a < \beta - 1$, from theorem (1.1) it follows that $y(a+1)$ and $y(a-1)$ both cannot be zero. If $y(a+1)y(a-1) \geq 0$, then (S_4) implies that y_n cannot have generalized zeros at both α and β, which is a contradiction. Thus, $y(a-1)y(a+1) < 0$.

Corollary 1.4. If y_n is a nontrivial solution of (1) with $y(\alpha) = y(a) = y(\beta) = 0$, where

$$\alpha < a < \beta - 1, \text{then } y(a+1) \neq 0.$$

Corollary 1.5. If a nontrivial solution y_n of theorem (1.1) has a zero at α and a generalized zero at β, where $\alpha < \beta$, then y_n cannot have consecutive zeros at $a, a+1$ where $\alpha < a < \beta - 1$.

Theorem 1.6. If two nontrivial solutions y_n and v_n of (1.1) have three zeros in common, then y_n and v_n are linearly dependent, i.e. specifying any three zeros uniquely determines a nontrivial solution up to a multiplicative constant.

Proof. If $y(\alpha) = y(a) = y(a+1) = v(\alpha) = v(a) = v(a+1) = 0$, for some α and a, where $0 \leq \alpha < a$, then by theorem 1.1, $u(a+2) \neq 0$ and $v(a+2) \neq 0$. Define $w(n) = v(a+2)y(n) - y(a+2)v(n)$. Since $w(n)$ is a linear combination of $y(n) and\ v(n)$, it is a solution of (1.1). However, $w(\alpha) = w(a) = w(a+1) = w(a+2) = 0, and\ so\ w(n)$ must be the trivial solution of (1.1) by theorem (1.1). Since $u(a+2)\ and\ v(a+2)$ are nonzero, $u(n) and\ v(n)$ must be constant multiples of each other.

Next, if $y(\alpha) = y(a) = y(\beta) = v(\alpha) = v(a) = v(\beta) = 0$, where $\alpha < a < \beta - 1$, then by corollary 1.5, $y(a+1) \neq 0\ and\ v(a+1) \neq 0$. Define $w(n) = v(a+1)y(n) - y(a+1)v(n)$.

Clearly, $w(\alpha) = w(a) = w(a+1) = w(\beta) = 0$, which contradicts corollary 1.4 unless$w(n) \equiv 0$.But this means $y(n)\ and\ v(n)$ are constant multiples of each other. This completes the proof.

Definition 1.7. A solution $y(n)$ of (1.1) is called recessive if there exists an $a \in N$ such that for all $k \in N(a)$.

$$y(n) > 0,\ \Delta y(n) \leq 0,\ \ \Delta^2 y(n) \geq 0\ and\ \ \Delta^3 y(n) \leq 0 \qquad \ldots(5)$$

Let $y^m(n)$ be the solution of (1.1) satisfying $y^m(m) = y^m(m+1) = y^m(m+2) = 0\ and\ y^m(0) = 1$ and where $m \in N(1)$. For each m,$y^m(n)$ exists and is unique. The existence is clear from theorem 1.1 and a normalization. While the uniqueness follows from theorem 1.6.Note that by construction.

$$0 \leq y^m(n) \leq 1\ \ \ for\ all\ \ \ k \in N(0, m+2) \qquad \ldots(6)$$

Also, Theorem (1.1) implies that

$$y^m(n) \geq y^m(n+1)\ \ \ \ for\ all\ k \in N \qquad \ldots(7)$$

We now consider m sequence $\{y^m(1)\}$. By (5), $0 \leq y^m(1) \leq 1$ for all $m \in N(1)$, thus

$\lim_{m\to\infty} sup\{y^m(1)\}$ exists, we call it $y(1)$. Then, there exists a subsequence $\{m_{1l}\} \subseteq N(1)$ such that

$$y^m(k+2)(p_{m+k}\Delta^2 y_{m+k}) + y^m(k+1)(q_{m+k}\Delta^2 y_{m+k}) = -r_{m+k}y_{m+k} \quad \ldots(8)$$

Consider (8) with $k = 2$ and m replaced by m_{3l}. we can conclude that $\lim_{l\to\infty} y^{m_{3l}}(5) = y(5)$. Proceeding inductively, we conclude that $\lim_{l\to\infty} y^{m_{3l}}(k) = y(k)$ exists for any $k \in N$.

Replacing m by m_{3l} in (8) and letting $l \to \infty$, we conclude that $y(k)$ is a solution of (1). Also,

$$y(k) \geq y(k+1) \geq 0 \quad \ldots(9)$$

This follows from (7) by replacing m by m_{3l}, fixing k, and letting $l \to \infty$. From (9) we conclude that

$$\lim_{k\to\infty} y(k) \text{ exists, and we shall call it L} \quad \ldots(10)$$

We will now show that this $y(k)$ is a recessive solution of (1).

Theorem 1.7. The solution $y(k)$ constructed above is a recessive solution of (1). In addition $\Delta y(k), \Delta^2 y(k)$ *and* $\Delta^3 y(k)$ all monotonically approach zero as $k \to \infty$.

Proof. We will first show that (5) is satisfied. By (7) and theorem 1.1, $y^{m_{3l}}(m_{3l} + 3) < 0$.

Choosing $m_{3l} \geq 3$ and using (S_3) with $a = m_{3l} + 1$, we can conclude that for any k such that $2 \leq k \leq m_{3l} + 1, \Delta y^{m_{3l}}(k-1) \leq 0, \Delta^2 y^{m_{3l}}(k-1) \geq 0$ and $\Delta^3 y^{m_{3l}}(k-1) \leq 0$.

Letting $l \to \infty$ implies that $y(k)$ satisfies (5) for a=1 and is recessive. We note that $y(k)$ also satisfies (5) for a=0. Concerning the monotonicity, we choose any $k \in N(2)$ and any $m_{3l} \geq k$.

Then, $\Delta^2 y^{m_{3l}}(k-1) \geq 0$ which means $\Delta y^{m_{3l}}(k) \geq \Delta y^{m_{3l}}(k-1)$, and hence $0 \leq -\Delta y^{m_{3l}}(k) \leq -\Delta y^{m_{3l}}(k-1)$. Taking the limit as $l \to \infty$ implies that $\Delta y(k)$ is monotonically decreasing in absolute value. By (1.1), Since $y(k)$ monotonically approaches a finite limit, $\Delta y(k) \to 0$ as $k \to \infty$. The argument that $\Delta^2 y(k)$ *and* $\Delta^3 y(k)$. monotonically approach zero is similar. By theorem 1.7 this recessive solution $y(k)$ of (1.1) can be return as –

$$\Delta^2(p_n\Delta^2 y_n) + q_{n+1}\Delta^2 y_{n+1} = l + \frac{1}{6}\sum_{l=k}^{\infty}(l-k+1)(l-k+2)$$
$$(l-k+3)r(l)y(l) \quad \ldots(11)$$

Corollary.1.8. If $\sum_1^{\infty} l^3 r(l) = \infty$, then the recessive solution $y(k)$ of (1.1) constructed above approaches zero as $k \to \infty$.

Corollary 1.9. Suppose that $y(k)$ *and* $v(k)$ are two recessive solutions of (1.1) such that $y(a) = v(a)$. If $y(k) \geq v(k)$ for all $k \in N(a)$, *then* $y(k) \equiv v(k)$.

Proof. Let $l = \lim_{k\to\infty} y(k)$ and $h = \lim_{k\to\infty} v(k)$. By hypothesis, $l \geq h$. Thus, if $w(k) = y(k) - v(k)$, than from (11) with $k = a + 2$ we have

$$0 \geq l - m + \frac{1}{6}\sum_{l=a+2}^{\infty}(l-a-1)(l-a)(l-a+1)r(l)w(l) \geq 0$$

From this we conclude that $y(k) = v(k)$.

CONCLUSION

The oscillatory properties of Fourth order Difference Equation become Oscillate.

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Accepted : 31.10.2016