Scientific Reviews & Chemical Communications

Sci. Revs. Chem. Commun.: 2(3), 2012, 264-271
ISSN 2277-2669

LAPLACE TRANSFORM IN FINANCE

N. A. PATIL* and VIJAYA N. PATIL[a]

Department of Applied Sciences and Humanities (Mathematics), Shri Sant Gajanan Maharaj College of Engineering, Shegaon, Dist. Buldhana – 444203 (M.S.) INDIA
[a]Department of Applied Sciences and Humanities (Mathematics), Padmashri Dr. V. B. Kolte College of Engineering, MALKAPUR Buldhana – 443101 (M.S.) INDIA

(*Received : 17.02.2012; Revised : 14.03.2012; Accepted : 23.03.2012*)

ABSTRACT

The main purpose of this note is to demonstrate how Laplace transforms techniques can be useful to economist. Laplace transforms have been widely used in valuing financial derivatives. In this paper, the present discounted value equation in finance is shown to be equivalent to the Laplace transformation in mathematics. Also we will focus on the general properties of the Laplace transform & present value rules for the elementary functions. And shows how the time derivative property is useful for the derivation of each rule using Laplace transforms/present value.

Key words: Present value, Cash flow, Consol, Time derivative, Laplace transform.

INTRODUCTION

During the last thirty years, the Laplace transforms has found an increasing number of applications in the fields of physics and technology. The transform itself has a very close relation to fundamental economic ideas.

The outline of this note is as follows –

In section 1 we introduce the Laplace transforms as present value rules.

In section 2 we identified the present value rules for each of the elementary functions.

In section 3 we discussed the general properties of Laplace transforms with present value rules.

In section 4 we show the application of time derivative property to each of the present value rules.

In section 5 we show how the simple rules can be combined to approximate the present value of complex cash flows.

Laplace transforms as a present value rules

The Laplace transform of a time variable functions g(t) for all *positive* values of *t*, is defined as-

---

Available online at www.sadgurupublications.com
*Author for correspondence; E-mail: napshegaon@rediffmail.com, napatil63@gmail.com, vijayaphirke@gmail.com

$$L\left[g(t)\right] = \int_0^\infty e^{-st}\ g(t)\ dt \qquad \ldots(1)$$

Provided that the integral exists,'s' is a parameter which may be real or complex number. Because of the upper limit in the integral is infinite, the integral on R.H.S. of equation (1) is an improper integral i.e.

$$\int_0^\infty e^{-st}\ g(t)\ dt = \lim_{T \to \infty} \int_0^T e^{-st}\ g(t)\ dt$$ and it is a function of 's'.

Therefore equation (1) can be written as -

$$L\left[g(t)\right] = G(s) = \int_0^\infty e^{-st} g(t)\ dt \qquad \ldots(2)$$

(Now we think about the present value problem in finance.)

One of the most recurring problems in finance is to estimate the present value of a given future cash flow. If the interest rate is constant & equal to 'θ' then the present value of a future cash flow, S(t) a function of t is given by:

$$P(t) = \sum_{t=1}^{\theta} \frac{S(t)}{(1+\theta)^t}$$

$$= \sum_{t=1}^{\theta} e^{-\theta t}\ S(t) \qquad \ldots(3)$$

Here the present value function is continuous and the time is bounded between 0 & N (some finite quantity). In the limiting case, replacing summation to an integral, the above equation can be expressed as,

$$P(\theta) = \int_0^N e^{-\theta t}\ S(t)\ dt \qquad \ldots(4)$$

(Due to presence of an integral, the domain of the computation changes from time 't' to rate 'θ' & therefore, the present value becomes a function of the rate, θ).

Now, if $N \to \infty$ then the equation (4) becomes:

$$P(\theta) = \int_0^\infty e^{-\theta t}\ S(t)\ dt \qquad \ldots(5)$$

Therefore, the present value of a future cash flow P (θ) is the Laplace transform of the cash flow S (t).

Present value rules for the elementary functions

From an economic point of view we immediately recognize the Laplace transform as the present value of the stream of returns S (t) at the interest rate θ.

Ex1. If the cash flow is $1, i.e. S(t)=1 the present value of a stream of return $1/year (a consol) at the interest rate θ is -

$$L[1]=\int_0^{\infty} e^{-\theta t}\ dt=\frac{1}{\theta}$$

This is the well known formula for the price of a consol bond or level payment stream.

[Verification: Let interest rate = 5% and cash flow = $100 then the present value of all the cash flows (summation over 100 yrs) would be equal to $ 1, 985 and by Laplace transform = $2,000]

Ex 2. The present value at interest rate ‘θ’ of a stream of returns growing at the rate α,

$$S(t)=e^{\alpha t} \text{ is } L\left[S(t)\right]=L\left[e^{\alpha t}\right]=\frac{1}{\theta-\alpha} \quad if\, \theta>\alpha$$

Rule 1 & 2 are the well known expressions for the present value of the consol & the geometric (or exponential) growth stream, resp. Rule 3 shows that an arithmetic (or linear) growth stream is equivalent to receiving one consol per period in perpetuity. This rule is used in many standard sources for present value solutions.Rule 4 is a power sequence which is natural extension of Rule 1 and 3.If we put n=0 in Rule 4, we get the equation for the present value of the consol (Rule 1) & if we put n=1 in 4, we get an arithmetic growth stream (Rule 3).

Table 1: Present values for the elementary functions

S. No.	Cash flow $S(t)$	Present values $P(\theta)$
1	Constant (1)	$\frac{1}{\theta}, \theta > 0$
2	Geometric $e^{\alpha t}$	$\frac{1}{\theta-\alpha}, \theta>\alpha$
3	Arithmetic *(t)*	$\frac{1}{\theta^2}, \theta>0$
4	Power (t^n)	$\frac{n!}{\theta^{n+1}}$
5	sine [sin t]	$\frac{1}{1+\theta^2}$
6	cosine [cos t]	$\frac{\theta}{1+\theta^2}$

Ex. suppose a continuous cash flow of a crop grows quadratically with time & given by –

$$S(t)=200*t^2$$

The present value of the set of infinite cash flows is found by,

Transform: Laplace (200*t^2, t, θ) is $\frac{400}{\theta^3}$

So if the yield is 10%, them the present value of all the crops is -

400/{(10%) ^3} = $400,000

Rule 5 & 6 shows that the present value rules for elementary periodic cash flows are neither less malleable (easily controlled) nor more troublesome than present value rules for elementary noncyclical cash flows. On the contrary, the rules are noticeable (i.e. easily seen or deleted) simply. At moderate rates of discount, the present value of the standard sine cash flow having unit amplitude is approximately $1 and for standard cosine cash flow is approximately equal to the rate of discount.

General properties of laplace transform

In table 1 we show a list of commonly used transforms. Such a table is certainly quite valuable but even more useful is the general properties of the Laplace transform which allow us to work with these transforms in an algebraic fashion.

(Let us look at some of the main properties)

(P_1): Linearity: The Laplace transform is linear operator; it allows us to deduce more complex transforms from simple transforms. (P_2) shows that scaling a cash flow by geometric growth term is equivalent to corresponding reduction in the rate of discount. Both rules are readily understood from the definition of the Laplace transform as the integral of an exponentially weighted function. (P_3) shows the effect of scaling a cash flow by the linear growth term.

[For the confirmation we recall the derivative of the exponential function, $e^{-\theta t}$ with respect to θ is simply the function itself scaled by $-t$.]

Table 2: The algebra of laplace transform/present value

P_1	Linearity	$\alpha S(t)+\beta R(t)$	$\alpha P(\theta)+\beta R(\theta)$
P_2	Geometric scaling	$e^{\alpha t}S(t)$	$P(\theta-\alpha),\ \theta>\alpha$
P_3	Multiplication by t	$tS(t)$	$-P'(\theta)$
P_4	Division by t	$S(t)/t$	$\int_{\theta}^{\infty} P(y)dy$
P_5	Time Shift	$S(\alpha+\beta t)$ for $t \geq \alpha/\beta$ 0 for $t < \alpha/\beta$	$e^{\theta\alpha/\beta}(\frac{1}{\beta})P(\theta/\beta)$
P_6	Point Flow	$S(t)$ for $t=k$ 0 for $t \neq k$	$e^{-\theta t}\ S(k)$
P_7	Finite life	$S(t)$ for $t \leq k$ 0 for $t \geq k$ With $S(t)= R(t-k)$	$P(\theta) - e^{\theta k}\ R(\theta)$
P_8	Time derivative	$S'(t)$	$\theta P(\theta) - S(0)$
P_9	Integral	$\int_0^t S(y)dy$	$P(\theta)/\theta$

(P_4), the division property is a corollary to (P_3) which obtained from Leibnitz's rule for the derivative of a definite integral taken with respect to its lower bound. (P_5) is the change of scale property, which is helpful for evaluating cash flows with altered time schedules. This property is also used in conjunction with the trivial rule for the transform of a single payment (P_6) to evaluate flows with the finite lives as indicated in (P_7).

(P_8) is the time derivative property which is the relation between Laplace transform for cash flows & their time derivatives. It has nontrivial solution. This property is different from the other rules of table-I and hence important for this note. All the six expressions of table-I can be derived by using this single property.

We can confirm this property (P_8) by using integration by parts for the function, $e^{-\theta t} S'(t)$.

(P_9) is a corollary to (P_8) which obtained from Leibnitz's rule for the derivative of definite integral taken with respect to its upper bound.

Applications of the time-derivative property on the present value rules

As we shown in section-1, the present value of a future cash flow P *(θ)* is the Laplace transform of the cash flow $S(t)$, i.e. $L[S(t)] = P(\theta)$

Using this notation rewrite the time derivative property (P_8) as -

$$L[S(t)] = \frac{S(0)}{\theta} + \frac{L[S'(t)]}{\theta} \qquad \ldots(6)$$

Now we apply this time derivative property to each of the present value rule.

The rule for the consol (I.1) follows trivially from equation (6).

For S (t) = 1 $\Rightarrow$ S (0) =1 & $S'(t)$ = 0, we get

$$L[S(t)] = \frac{1}{\theta}$$

In turn, the consol rule can be define as each asset is valued as if its cash flow were projected at a constant level equal to the current rate plus the present value of the time derivative of the cash flow.

For geometric cash flow (2.I)

$S(t) = e^{\alpha t} \Rightarrow S(0) = 1$ & $S'(t) = \alpha\, e^{\alpha t}$, we get

$$L[e^{\alpha t}] = \frac{1}{\theta} + \frac{\alpha L\left[e^{\alpha t}\right]}{\theta} \qquad \ldots(7)$$

Alternatively, we could combine the consol rule with geometric scaling (P_2) to derive the rule for geometric growth.

For $S(t) = t \Rightarrow S(0) = 0$ & $S'(t) = 1$

And (I.3) follows immediately. Alternatively, we could combine the consol rule with (P_3) to derive the rule for linear growth.

For any power sequence, $S(t) = t^n$

$\Rightarrow S(0) = 0$ & $S'(t) = n\, t^{n-1}$

$$(6) \Rightarrow L\left[t^n\right] = \frac{n}{\theta} L\left[t^{n-1}\right] \qquad \ldots(8)$$

By repeating this observation n times we get the rule for power cash flow (I.4) alternatively, we can confirm the result by repeated application of (P_3).

For $S(t) = \sin t$, for $S(t) = \cos t$,

$\Rightarrow S(0) = 0$ $\quad\quad$ $\Rightarrow S(0) = 1$

$\Rightarrow S'(t) = \cos t$ $\quad\quad$ $\Rightarrow S'(t) = -\sin t$

$$\therefore (6) \Rightarrow L[\sin t] = \frac{L[\cos t]}{\theta} \quad ..... (9) \qquad L[\cos t] = \frac{1}{\theta} - \frac{L[\sin t]}{\theta} \qquad ...(10)$$

Solving equations (9) and (10), we get the rules for the simple periodic cash flows.

***All this combinations tabulated as follows:**

Cash Flow $L[s(t)]$	**Present discounted Value** ***P(θ)***	**Time derivative property** $L[S(t)] = \frac{S(0)}{\theta} + \frac{L[S'(t)]}{\theta}$	**Combination**
1	$\frac{1}{\theta}$	$\frac{1}{\theta}$	consol rule-1
$e^{\alpha t}$	$\frac{1}{\theta - \alpha} \text{ if } \theta > \alpha$	$\frac{1}{\theta} + \frac{\alpha}{\theta} L\left[e^{\alpha t}\right]$	consol rule-1+P_2
t	$\frac{1}{\theta^2}$	$\frac{L[1]}{\theta}$	Consol + P_3
t^n	$\frac{n!}{\theta^{n+1}}$	$\frac{n}{\theta} L[t^{(n-1)}]$	Consol +repeated application of P_3
Sin t	$\frac{1}{1+\theta^2}$	$\frac{L[\cos t]}{\theta}$	Consol +6.I
Cos t	$\frac{\theta}{1+\theta^2}$	$\frac{1}{\theta} - \frac{L[\sin t]}{\theta}$	Consol +5.I

Present value rules for complex cash flows

Now we consider the case of the lowly k-period annuity by using the above potential combinations.

We apply (P_7) with $S(t) = S(t-k) = 1$, we get

$$L[S(t-k)] = L[S(t)] = P(\theta) = \int_0^\infty S(t-k)\, e^{-\theta t}\, dt = \int_0^\infty (1)\, e^{-\theta t}\, dt = \frac{1}{\theta} = R(\theta)$$

And we get the following rule for the present value of an annuity,

$$P(\theta) = \left(1 - e^{-\theta k}\right) \frac{1}{\theta} \qquad ...(11)$$

This equation assumes that the 1st payment of the annuity is made at the end of the 1st time period. If instead the payments are made at the beginning of each time period, then the present value calculation would be similar to the above.

Equation (11) represents the finite level payment stream. It is the difference between payment beginning today & the payment differed from k-periods.

If we combine the Rule 5(T-I) with property (P_1) & (P_5) for $S(t) = \sum_i \alpha_i \sin(\beta_i + \gamma_i t)$, we get the present value of the complex cyclical cash flow as follows.

$$P(\theta) = \sum_i \alpha_i \, e^{\left(\theta\beta_i/\gamma_i\right)} \frac{\gamma_i}{\gamma_i^2 + \theta_i^2} \qquad \ldots(12)$$

In this example, the cash flow is measured by the amplitude *(a_i)* & frequency *(b_i)* of each sine value so, there is certainty (predictability) about the future cash flow or rate of discount. There is no variation in the cash flow as having substantial impact on the present value of the cash flow.

CONCLUSION

In this article we have presented the close relationship between Laplace transform & present discounted value in finance. The result seems to be new & to have a potential to increase the practical utility of Laplace transform especially in finance. The transforms are considered as a tool to make mathematical calculations easier. However it is important to notice that frequency domain is possible appreciate also in the real world & applied in the areas like economics or finance.

A list of combination of Rule in table-I and property in table-II is quite large as compare to extensive tables of Laplace transform. Many of the individual entries in these lists are of unsure practical value in finance. There is no collection of rules could serve as a valuable reference. But the Laplace transform is the big source for present discounted value function to illustrate the enhanced problems.

REFERENCES

1. Roy S Freedman, Introduction to Financial Technology, Academic Press, 24-Apr., Business & Economics, 130-135 (2006).
2. Duncan K. Foley, A note for Laplace Transforms.
3. Eugene F. Brigham, Joel F. Houston, Fundamentals of Financial Management, 130-140.
4. D. V. Widder, What is the Laplace Transform?", Harvard University (Princeton University Press (1946).
5. Stephen A Buser, (March 1986), Laplace Transforms as present Value Rules, J. Finance, Vol. XLI, No. 1.
6. Risk Latte, Laplace Transforms & the Time Value of Money-I, Dec 25 (2010).
7. E. Kreyszig, Advanced Engineering Mathematics, 8th Edition, 251-279.
8. M. T. Tham, Deptt. of Chemical & Process Engineering, Process Dynamics, Laplace Transforms, University of Newcastle upon Tyne (Oct. 1999).
9. Robert W. Grubbstrom, On the Application of the Laplace Transform to Certain Economic Problems, Management Science, **13(7)**, Series A, Sciences (March-1967)
10. William McC. Siebert, Circuits, Signals & system, Vol. **2**, 43-48.

11. Ben J. Heijdra, Fiscal Policy Multipliers and Finite Lives, Mimeo, University of Groningen (1999).
12. Ben J. Heijdra, Using the Laplace Transform for Dynamic Policy Analysis, Nov. (1999). http://www.eco.rug.nl/medewerk/heijdra/download.htm
13. Dean G. Duffy, Advanced Engineering Mathematics, 161-169.
14. Rajesh Pandey, A text book of Engineering Mathematics, Vol.-I.
15. D. T. Deshmukh, Advanced Modern Engineering Mathematics, 921-960.
16. Joel L. Schiff, The Laplace Transform: Theory & applications, Auckland University, Springier, 1-30.
17. Peter V. O'neil, Advance Engineering Mathematics, Seventh Edition, University of Alabama of Birmingham, 77-80.
18. Hans-Joachim Girlich, a Journal of finance, Between Paris and Linkoping: Applications of Integral Transforms to Planning and Finance, University of Leipzig, Germany.
19. M. R. Spiegel, Laplace transform, McGraw-Hill, New York (1965).