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Kalman filtering without direct feedthrough from unknown inputs

Jilai Liu[1], Shuwen Pan[2]*, Yanjun Li[2]

[1]Zhejiang Provincial Education Examination Authority, Hangzhou 310012, (P.R.CHINA)

[2]Key Discipline of Automation, School of Information and Electrical Engineering, Zhejiang University City College, Hangzhou, 310015, (P.R.CHINA)

E-mail : pansw@zucc.edu.cn

ABSTRACT

The problem of joint input and state estimation is addressed in this paper for discrete-time stochastic systems without direct feedthrough from unknown inputs to outputs. Following the identical idea of previous study on discrete-time stochastic systems with direct feedthrough, the weighted least squares estimation for an extended state vector including unknown inputs and states is used to derive a Kalman filter with unknown inputs without directfeedthrough (KF-UI-WDF) approach. The information on unknown inputs is not needed for KF-UI-WDF and the necessary and sufficient conditions for the state and input detectability are presented. The estimators of KF-UI-WDF are proven minimum variance unbiased (MVU) ones.

KEYWORDS

Kalman filtering; Estimation; Minimum variance; Unbiased filter; Least squares estimation.

INTRODUCTION

When unknown inputs are presented in stochastic linear systems, traditional filter approaches for state estimation are not feasible any more. To solve this problem, two different categories of filter approaches have been studied, i.e., (i) the augmented state Kalman filters (ASKFs) [1-3]; and (ii) the robust Kalman filters (RKFs) [4-9]. The former category (ASKFs) requires the dynamical information of unknown inputs, which is hard to be obtained in reality. Different from ASKFs, the latter category (RKFs) doesn't need any prior information of unknown inputs. As the result of the feasibility, RKFs attracted more attention of researchers than ASKFs in the past decade. To the best of the authors' knowledge, the latest progress on RKFs is made by Gillijns and De Moor [10,11] for discrete-time system without and with direct feedthrough, respectively. In Gillijns and De Moor [10,11], the unknown input estimator is proven an minimum variance unbiased (MVU) estimator, which extended the results of Kitanidis [4], Darouach and Zasadzinski [5], and Hsieh [7], etc. However, the optimalities for the unknown input and state estimators in Gillijns and De Moor [10,11] are not achieved globally. For instance, the optimal gain matrices for the state estimation and input estimation are considered separately.

To obtain a globally optimal filter for the unknown input and state estimation, it is beneficial to check the derivation of traditional Kalman filter [12,13] for reference. The recursive solutions of traditional Kalman filter can be derived by using the weighted least squares estimation (LSE) for an unknown state vector with fixed dimension. Following the idea of weighted LSE with the aid of decomposing method [14-16] and collecting the unknown states and inputs in one unknown extended state vector with increasing dimension, Pan et al [17] proposes a minimum variance unbiased (MVU) filter for the estimation of unknown states and unknown inputs referred to as Kalman filter with unknown inputs (KF-UI), which is for discrete-time stochastic systems with direct feedthrough from unknown inputs to outputs. The recursive solutions of KF-UI are the direct extension of traditional Kalman filter and no heavier computational burden and prior information of unknown inputs are required.

For the case with direct feedthrough from unknown inputs to outputs, the measurements at $t = k\Delta\mathrm{t}$, $\mathbf{y}_k$ obviously possesses the information of the unknown inputs at $t = k\Delta t$, $\mathbf{d}_k$. On the other hand, for the case without direct feedthrough, the unknown inputs $t = (k-1)\Delta t$, $\mathbf{d}_{k-1}$, instead of $\mathbf{d}_k$, is reflected by $\mathbf{y}_k$. In this situation, the unknown inputs have to be estimated with one step delay, i.e., the estimation of $\mathbf{d}_{k-1}$ should be obtained no earlier than the time instant $t = k\Delta t$ [9,10]. Due to the time delay of unknown inputs, the filter solutions for the case with direct feedthrough are enormously different from the ones for the case without direct feedthrough. In this regard, the globally optimal filter for the case without direct feedthrough is derived in this paper following the samilar procedure described in [17]. As the KF-UI provided by Pan et al [17], the filter approach proposed herein is also a direct extension of traditional Kalman filter with low computational burden and without the requirements about prior information and the type of unknown inputs.

This paper is outlined in the following. First, the problem formulation is given in Section 2. The recursive solutions for extended state vector including unknown inputs and states are presented in Section 3. Then, the optimal filter for unknown inputs and states is proposed in Section 4. Furthermore, the proofs for minimum variance unbiased (MVU) estimators and their error covariance matrices are given in Section 5. Finally, Section 6 is the conclusion.

PROBLEM FORMULATION

Consider the following stochastic linear discrete-time system,

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{d}_k + \mathbf{w}_k \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{v}_k \tag{2}$$

where $\mathbf{x}_k \in R^n$ is the state vector; $\mathbf{d}_k \in R^m$ is the unknown vector, respectively; $\mathbf{y}_k \in R^p$ is the output measurement vector; the process noise $\mathbf{w}_k \in R^n$ and the measurement noise $\mathbf{v}_k \in R^p$ are mutually uncorrelated, zero-mean, white random signals with known covariance matrices, $\mathbf{Q}_k = E[\mathbf{w}_k \mathbf{w}_k^{\mathrm{T}}]$ and $\mathbf{R}_k = E[\mathbf{v}_k \mathbf{v}_k^{\mathrm{T}}]$, respectively. Unlike the case with direct feedthrough [17], it is observed from Eqs.(1) and (2) that the information of unknown inputs $\mathbf{d}_{k-1}$ instead of $\mathbf{d}_k$ is contained in the measurements $\mathbf{y}_k$. The unknown inputs can be any type of signals and no prior information of unknown inputs is given or assumed. In what follows, the bold face letter represents either a vector or a matrix.

Let $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ be the estimates of $\mathbf{x}_k$ and $\mathbf{d}_{k-1}$ at $t = k\Delta t$, respectively, given the observations ($\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_k$). Following the derivation of traditional Kalman filter [12], $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ can be determined by minimizing the objective function of sum square error as follows,

$$J_k = \sum_{i=1}^{k} \boldsymbol{\Delta}_i^{\mathrm{T}} \tilde{\mathbf{R}}_{i|k}^{-1} \boldsymbol{\Delta}_i = \bar{\boldsymbol{\Delta}}_k^{\mathrm{T}} \mathbf{W}_k \bar{\boldsymbol{\Delta}}_k \tag{3}$$

in which $\boldsymbol{\Delta}_i$ is the output error at $t = i\Delta t$ (i = 1, 2, …, k)

To make the unknown quantities involve only $\mathbf{x}_k$ and $\mathbf{d}_i$, the unknown quantities $\mathbf{x}_i$ (i = 1, 2, …, k–1) in Eq.(3) can be treated by expressing $\mathbf{x}_i$ in terms of $\mathbf{x}_k$ and $\mathbf{d}_j$ (j = i, i+1, …, k–1) through repeated applications of the transition relation based on state equation Eq.(1). Substituting the result into t Eq.(4) leads to the following expression for $\bar{\boldsymbol{\Delta}}_k$,

$$\bar{\boldsymbol{\Delta}}_k = \mathbf{Y}_k - \mathbf{A}_{e,k}\mathbf{x}_{e,k} \tag{4}$$

where $\mathbf{x}_{e,k} = \left[\mathbf{x}_k^{\mathrm{T}} \;\vdots\; \mathbf{d}_1^{\mathrm{T}} \;\cdots\; \mathbf{d}_{k-1}^{\mathrm{T}} \right]^{\mathrm{T}}$, $\mathbf{Y}_k$ and $\mathbf{A}_{e,k}$ are known matrices.

THE RECURSIVE SOLUTIONS FOR EXTENDED STATE VECTOR

As observed from Eqs.(3) and (4), the objective function J_k is a quadratic function of the unknown extended state vector $\mathbf{x}_{e,k}$. Assuming that the dimension of the output measurement vector $\mathbf{y}_k$ is greater than the dimension of unknown inputs $\mathbf{d}_{k-1}$, i.e., $p > m$, the least squares estimation of $\mathbf{x}_{e,k}$ at $t = k\Delta t$, $\hat{\mathbf{x}}_{e,k|k}$, is obtained by minimizing J_k in Eq.(3) as follows,

$$\hat{\mathbf{x}}_{e,k|k} = \mathbf{P}_{e,k}\left[\mathbf{A}_{e,k}^{\mathrm{T}}\mathbf{W}_k\mathbf{Y}_k\right] \tag{5a}$$

$$\mathbf{P}_{e,k} = \left[\mathbf{A}_{e,k}^{\mathrm{T}}\mathbf{W}_k\mathbf{A}_{e,k}\right]^{-1} \tag{5b}$$

where $\hat{\mathbf{x}}_{e,k|k}^{\mathrm{T}} = \left[\hat{\mathbf{x}}_{k|k}^{\mathrm{T}} \;\vdots\; \hat{\mathbf{d}}_{1|k}^{\mathrm{T}} \;\vdots\; \hat{\mathbf{d}}_{2|k}^{\mathrm{T}} \;\vdots\; \cdots \;\vdots\; \hat{\mathbf{d}}_{k-1|k}^{\mathrm{T}} \right]^{\mathrm{T}}$ is the estimation of $\mathbf{x}_{e,k}$, given the observations ($\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_k$).

From the solution of $\hat{\mathbf{x}}_{e,k|k}$ given by Eqs.(5a)&(5b), we obtain the recursive solutions for $\hat{\mathbf{x}}_{e,k|k}$ with the method of [14],

$$\hat{\mathbf{x}}_{e,k|k} = \begin{bmatrix} \bar{\mathbf{x}}_{e,k} + \bar{\mathbf{P}}_{e,k}\tilde{\mathbf{P}}_{e,k}^{-1}\hat{\mathbf{G}}_{k-1}\hat{\mathbf{d}}_{k-1|k} \\ \hdashline \hat{\mathbf{d}}_{k-1|k} \end{bmatrix} \tag{6}$$

where

$$\hat{\mathbf{d}}_{k-1|k} = -\mathbf{S}_k\hat{\mathbf{G}}_{k-1}^{\mathrm{T}}\tilde{\mathbf{P}}_{e,k}^{-1}(\boldsymbol{\Phi}_{k,k-1}^{*}\hat{\mathbf{x}}_{e,k-1|k-1} + \tilde{\mathbf{F}}_{k-1} - \bar{\mathbf{x}}_{e,k}) \tag{7}$$

In Eqs.(6)-(7), the estimate of the extended state vector at $t = k\Delta t$, $\hat{\mathbf{x}}_{e,k-1|k-1}$ is obtained from Eqs. (5a)&(5b) by replacing k with k–1.

THE RECURSIVE SOLUTIONS FOR STATE AND UNKNOWN INPUT ESTIMATORS

Since the extended state vector $\hat{\mathbf{x}}_{e,k|k}$ has an increasing dimension as k increases [14], the computational burden is quite heavy even for the recursive solutions presented in Eqs.(6)-(7). To obtain the analytical recursive solutions of $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$, the recursive solutions of $\hat{\mathbf{x}}_{e,k|k}$ in Eqs.(6)-(7) should be further decomposed. Finally, the recursive solutions for $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ are obtained with the method presented in [14]:

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_{\mathbf{x},k}(\mathbf{y}_k - \mathbf{C}_k \hat{\mathbf{x}}_{k|k-1}) \tag{8}$$

$$\hat{\mathbf{d}}_{k-1|k} = \mathbf{S}_k \mathbf{G}_{k-1}^{\mathrm{T}} \mathbf{C}_k^{\mathrm{T}} \mathbf{R}_k^{-1} (\mathbf{I}_p - \mathbf{C}_k \mathbf{K}_{\mathbf{x},k})(\mathbf{y}_k - \mathbf{C}_k \mathbf{A}_{k-1} \hat{\mathbf{x}}_{k-1|k-1}) \tag{9}$$

where

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{A}_{k-1} \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{G}_{k-1} \hat{\mathbf{d}}_{k-1|k} \tag{10}$$

$$\mathbf{S}_k = [\mathbf{G}_{k-1}^{\mathrm{T}} \mathbf{C}_k^{\mathrm{T}} \mathbf{R}_k^{-1} (\mathbf{I}_p - \mathbf{C}_k \mathbf{K}_{\mathbf{x},k}) \mathbf{C}_k \mathbf{G}_{k-1}]^{-1} \tag{11}$$

In Eqs.(8), (9) and (11), $\mathbf{K}_{\mathbf{x},k}$ is the Kalman gain matrix given by

$$\mathbf{K}_{\mathbf{x},k} = \mathbf{P}_{\mathbf{x},k|k-1} \mathbf{C}_k^{\mathrm{T}} (\mathbf{R}_k + \mathbf{C}_k \mathbf{P}_{\mathbf{x},k|k-1} \mathbf{C}_k^{\mathrm{T}})^{-1} \tag{12}$$

where $\mathbf{P}_{\mathbf{x},k|k-1}$ is given by

$$\mathbf{P}_{\mathbf{x},k|k-1} = \mathbf{A}_{k-1} \mathbf{P}_{\mathbf{x},k-1|k-1} \mathbf{A}_{k-1}^{\mathrm{T}} + \mathbf{Q}_{k-1} \tag{13}$$

where $\mathbf{Q}_{k-1}$ is the covariance matrix of the model noise vector $\mathbf{w}_{k-1}$, and $\mathbf{P}_{\mathbf{x},k-1|k-1}$ is given by

$$\mathbf{P}_{\mathbf{x},k-1|k-1} = (\mathbf{I}_n - \mathbf{K}_{\mathbf{x},k-1} \mathbf{C}_{k-1}) [\mathbf{P}_{\mathbf{x},k-1|k-2} + \mathbf{G}_{k-2} \mathbf{S}_{k-1} \mathbf{G}_{k-2}^{\mathrm{T}} (\mathbf{I}_n - \mathbf{K}_{\mathbf{x},k-1} \mathbf{C}_{k-1})^{\mathrm{T}}] \tag{14}$$

where $\mathbf{S}_{k-1}$, $\mathbf{K}_{\mathbf{x},k-1}$ and $\mathbf{P}_{\mathbf{x},k-1|k-2}$ are obtained from Eqs.(11)-(13) by replacing k by k–1, respectively. As the components of $\hat{\mathbf{x}}_{e,k|k}$ given by Eq.(5*a*), the state and unknown input estimation $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ achieve optimal values simultaneously when $\hat{\mathbf{x}}_{e,k|k}$ becomes optimal. The description of the derivation procedure shows clearly that, the solutions of the new filter presented in Eqs.(8)-(14) is a direct extension of traditional Kalman filter [12]. To avoid the confusion with Kalman filter with unknown inputs (KF-UI) proposed in [17], the new filter presented in Eqs.(8)-(14) in this paper is referred to as the Kalman filter with unknown inputs without direct feedthrough (KF-UI-WDF), which is not available in the previous literatures. When all the inputs are known, i.e., $\mathbf{G}_k \mathbf{d}_k = \mathbf{0}$ in Eq.(1), the proposed KF-UI-WDF reduces to the traditional Kalmen filter [12].

THE DISCUSSION ABOUT PROPERTIES OF THE ESTIMATORS

(1). $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ are minimum variance unbiased (MVU) estimators.

Proof

Following the unbiasedness proof [12], the <u>unbiased</u> estimator of $\mathbf{x}_{e,k}$ can be expressed by

$$\hat{\mathbf{x}}_{e,k|k} = \sum_{i=1}^{k} \tilde{\mathbf{S}}_i \bar{\mathbf{y}}_{i|k} = \bar{\mathbf{S}}_k \mathbf{Y}_k \tag{15}$$

where $\tilde{\mathbf{S}}_{\mathrm{i}}$ = $[(n + m(k–1))\times p]$ matrix, $\bar{\mathbf{S}}_k = [\tilde{\mathbf{S}}_1 \mid \tilde{\mathbf{S}}_2 \mid \cdots \mid \tilde{\mathbf{S}}_k]$. Using $E\,[\hat{\mathbf{x}}_{e,k|k}] = \mathbf{x}_{e,k}$ with the aid of Eq.(5*a*), one obtains

$$\begin{aligned} &\mathbf{x}_{e,k} = E\,[\hat{\mathbf{x}}_{e,k|k}] = \bar{\mathbf{S}}_k E\,[\mathbf{Y}_k] = \bar{\mathbf{S}}_k \mathbf{A}_{e,k+1} \mathbf{x}_{e,k} \\ &\Rightarrow [\bar{\mathbf{S}}_k \mathbf{A}_{e,k+1} - \mathbf{I}_{n+m(k-1)}]\mathbf{x}_{e,k} = \mathbf{0} \end{aligned} \tag{16}$$

Since $\mathbf{x}_{e,k} \neq \mathbf{0}$, from Eq.(16), we have

$$\bar{\mathbf{S}}_k \mathbf{A}_{e,k+1} - \mathbf{I}_{n+m(k-1)} = \mathbf{0} \tag{17}$$

With the aid of above results, we have

$$E\,[(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})^{\mathrm{T}}] = E\,(\bar{\mathbf{S}}_k \mathbf{W}_k^{-1} \bar{\mathbf{S}}_k^T) \tag{18}$$

in which $\mathbf{W}_k$ is givne by Eq.(3). Using the following relation,

$$(\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} + [\,\bar{\mathbf{S}}_{\mathrm{k}} - (\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{e,k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}}\,] = \bar{\mathbf{S}}_{\mathrm{k}} \tag{19}$$

Eq.(18) becomes

$$\begin{aligned} &\mathrm{E}\,[(\mathbf{x}_{\mathrm{e,k}} - \hat{\mathbf{x}}_{\mathrm{e,k|k}})(\mathbf{x}_{\mathrm{e,k}} - \hat{\mathbf{x}}_{\mathrm{e,k|k}})^{\mathrm{T}}] \\ &= \mathrm{E}\left\{(\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} + [\,\bar{\mathbf{S}}_{\mathrm{k}} - (\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{e,k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}}\,]\right\} \mathbf{W}_{\mathrm{k}}^{-1} \\ &\quad \left\{(\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}} + [\,\bar{\mathbf{S}}_{\mathrm{k}} - (\mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{e,k}} \mathbf{A}_{\mathrm{e,k}})^{-1} \mathbf{A}_{\mathrm{e,k}}^{\mathrm{T}} \mathbf{W}_{\mathrm{k}}\,]\right\}^{\mathrm{T}} \end{aligned} \tag{20}$$

Substituting Eq.(19) into Eq.(20) and minimizing the trace of the resulting $E\,[(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})^{\mathrm{T}}]$, one obtains the optimal $\bar{\mathbf{S}}_k$, i.e.,

$$\bar{\mathbf{S}}_{k,optimal} = (\mathbf{A}_{e,k}^{\mathrm{T}} \mathbf{W}_k \mathbf{A}_{e,k})^{-1} \mathbf{A}_{e,k}^{\mathrm{T}} \mathbf{W}_k \tag{21}$$

As a result, the minimum variance unbiased (MVU) estimator of $\mathbf{x}_{e,k}$ is given by

$$\hat{\mathbf{x}}_{e,k|k} = \bar{\mathbf{S}}_{k,optimal} \mathbf{Y}_k = (\mathbf{A}_{e,k}^{T} \mathbf{W}_k \mathbf{A}_{e,k})^{-1} \mathbf{A}_{e,k}^{\mathrm{T}} \mathbf{W}_k \mathbf{Y}_k \tag{22}$$

Eq.(22) is identical to Eq.(5a). Therefore, we conclude that the least squares estimator $\hat{\mathbf{x}}_{e,k|k}$ for $\mathbf{x}_{e,k}$ is also the minimum variance unbiased (MVU) one. Comparing the structures of $\mathbf{x}_{e,k}$ and $\hat{\mathbf{x}}_{e,k|k}$ leads to the conclusion that $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{d}}_{k-1|k}$ given by Eqs.(8) and (9) are minimum variance unbiased (MVU) estimators of $\mathbf{x}_k$ and $\mathbf{d}_{k-1}$.

The proof is completed.

(2) Among the recursive solutions presented in Eqs.(8)-(14), $\mathbf{P}_{\mathbf{x},k|k-1}$, $\mathbf{P}_{\mathbf{x},k-1|k-1}$, and $\mathbf{S}_k$ are the error covariance matrices of $\hat{\mathbf{x}}_{k|k-1}$, $\hat{\mathbf{x}}_{k-1|k-1}$ and $\hat{\mathbf{d}}_{k-1|k}$, respectively, i.e.,

(i) $\mathbf{P}_{\mathbf{x},k-1|k-1} = E\,[(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})^{\mathrm{T}}]$

(ii) $\mathbf{P}_{\mathbf{x},k|k-1} = E\,[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})^{\mathrm{T}}]$;

(iii) $\mathbf{S}_k = E\,[(\mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1|k})(\mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1|k})^{\mathrm{T}}]$.

Proof

For (i) and (iiii):

Following the derivation for traditional Kalman filter [12], the estimation error $\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k}$ can be expressed as follows by using Eqs.(4) and (5*a*), i.e.,

$$\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k} = -(\mathbf{A}_{e,k}^{\mathrm{T}}\mathbf{W}_k\mathbf{A}_{e,k})^{-1}\mathbf{A}_{e,k}^{\mathrm{T}}\mathbf{W}_k\mathbf{E}_k \quad \textbf{(23)}$$

Substituting Eq.(23) into $E\,[(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})^{\mathrm{T}}]$ and using the definition of $\mathbf{W}_k$ given by Eq.(3), the error covariance matrix can be obtained as

$$E\,[(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})^{\mathrm{T}}] = \mathbf{P}_{e,k} \quad \textbf{(24)}$$

where $\mathbf{P}_{e,k}$ is given by Eq.(5b). It is observed from the structures of $\mathbf{x}_{e,k}$ and $\hat{\mathbf{x}}_{e,k|k}$ that the top left ($n\times n$) and bottom right ($m\times m$) sub-matrices of $E\,[(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})(\mathbf{x}_{e,k} - \hat{\mathbf{x}}_{e,k|k})^{\mathrm{T}}]$ are $E\,[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^{\mathrm{T}}]$ and $E\,[(\mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1|k})(\mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1|k})^{\mathrm{T}}]$, respectively.

$\mathbf{P}_{\mathbf{x},k-1|k-1}$ is then obtained by replacing *k* with *k*–1 in $\mathbf{P}_{\mathbf{x},k|k} = E\,[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^{\mathrm{T}}]$.

For (ii):

It is observed from the relations between $\mathbf{x}_{e,k}$ and $\mathbf{x}_{e,k-1}$ that

$$\tilde{\mathbf{x}}_{e,k} = \mathbf{\Phi}^{*}_{k,k-1}\mathbf{x}_{e,k-1} + \tilde{\mathbf{w}}_{k-1};\ \hat{\tilde{\mathbf{x}}}_{e,k} = \mathbf{\Phi}^{*}_{k,k-1}\hat{\mathbf{x}}_{e,k-1|k-1} \quad \textbf{(25)}$$

where

$$\tilde{\mathbf{x}}_{e,k} = [\mathbf{x}_k^{\mathrm{T}} \mid \mathbf{d}_1^{\mathrm{T}} \mid \cdots \mid \mathbf{d}_{k-2}^{\mathrm{T}}]^{\mathrm{T}} \text{ and } \hat{\tilde{\mathbf{x}}}_{e,k} = [\hat{\mathbf{x}}_{k|k-1}^{\mathrm{T}} \mid \hat{\mathbf{d}}_{1|k-1}^{\mathrm{T}} \mid \cdots \mid \hat{\mathbf{d}}_{k-2|k-1}^{\mathrm{T}}]^{\mathrm{T}}.$$

In Eq.(25), $\tilde{\mathbf{w}}_{k-1} = [\mathbf{w}_{k-1}^{T} \mid \mathbf{0}_{1\times m} \mid \cdots \mid \mathbf{0}_{1\times m}]^{\mathrm{T}}$ is a [*n*+*m*(*k*–2)]-vector. Therefore,

$$\tilde{\mathbf{x}}_{\mathrm{e,k}} - \hat{\tilde{\mathbf{x}}}_{\mathrm{e,k}} = \mathbf{\Phi}^{*}_{\mathrm{k,k-1}}(\mathbf{x}_{\mathrm{e,k-1}} - \hat{\mathbf{x}}_{\mathrm{e,k-1|k-1}}) + \tilde{\mathbf{w}}_{\mathrm{k-1}} \quad \textbf{(26)}$$

Taking the expectation of the both sides of Eq.(26), one obtains

$$E[(\tilde{\mathbf{x}}_{e,k} - \hat{\tilde{\mathbf{x}}}_{e,k})\,(\tilde{\mathbf{x}}_{e,k} - \hat{\tilde{\mathbf{x}}}_{e,k})^{\mathrm{T}}] = \tilde{\mathbf{P}}_{e,k} \quad \textbf{(27)}$$

As a result, one obtains $\mathbf{P}_{\mathbf{x},k|k-1} = E\,[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})^{\mathrm{T}}]$. The proof is completed.

CONCLUSIONS

A globally optimal filter referred to as Kalman filter with unknown inputs without directfeedthrough (KF-UI-WDF) is proposed for the state and unknown input estimation of discrete-time stochastic systems without direct feedthrough. The analytical recursive solutions of KF-UI-WDF are derived with the weighted least squares estimation of an extended state vector including states and unknown inputs. The unknown inputs can be any type of signals and prior information of unknown inputs is not required. The estimators of KF-UI-WDF are proven minimal variance unbiased (MUV) ones and necessary and sufficient conditions for the uniqueness of the filter solutions are presented.

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