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GENERATING FUNCTIONS FOR THE ULTRASPHERICAL POLYNOMIALS BY TRUESDELL METHOD

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ABSTRACT

Ultraspherical polynomials $C_n^{\upsilon}(x)$ (Gegenbaur polynomials) are one of the most important families of orthogonal polynomials in mathematical chemistry, mathematical physics, probability theory, differential equations, combinatorics etc. One of the simplest ways to construct them is through their generating functions. In this paper, the generating functions for ultraspherical polynomials $C_n^{\upsilon}(x)$ are obtained by using the Truesdell's method giving a suitable interpretation to the index *n*. Further, a pair of linearly independent differential recurrence relations are used in order to derive generating functions for $C_n^{\upsilon}(x)$. The principal interest in our results lies in the fact that, how the Truesdell's method is utilized in an effective and suitable way to Gegenbaur polynomials in order to derive two generating functions independently from ascending and descending recurrence relations, respectively. The generating functions, in turn yield, the Legendre polynomials as special case for $\upsilon = \frac{1}{2}$. The results are well known in the theory of special functions. Mathematics Subject Classification (2010): Primary 33C10, secondary 33C45, 33C80.

Key words: Special functions, Ultraspherical polynomials, Generating functions.

INTRODUCTION

Generating functions play an important role in the investigation of various useful properties of the sequences, which they generate. They are used with good effect for the determination of the asymptotic behavior of the generated sequence $\{f_n\}_{n=0}^{\infty}$ as $n \to \infty$. In recent years, the development of advanced computers has made it necessary to study the hypergeometric polynomials with series representations from the numerical point of view[1]. Because of the important role which hypergeometric polynomials play important role in problems of applied mathematics, the theory of generating functions has been developed

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various directions and found wide applications in different branches of science and technology.

Many authors[2-7] studied the generating functions for the generalized hypergeometric polynomials by various methods. The generalized hypergeometric polynomials have many applications in different branches of analysis, such as harmonic analysis, quantum physics, molecular chemistry, number theory etc. A few of them, for example, the ultraspherical polynomials appear naturally as extensions of Legendre polynomials in the content of potential theory and harmonic analysis. The Newtonian potential in R^n has the expansion, valid with $\upsilon = (n-2)/2$,

$$\frac{1}{|x-y|^{n-2}} = \frac{|x|^k}{|y|^{k+n-2}} C^{\upsilon}_{n,k}(x,y)$$

Also, in the study of oscillations and waves, sine and cosine functions play a central role. They come from the solutions of the wave (Helmholtz) equation in cartesian coordinates with the appropriate boundary conditions. They also form a basis for representing general waves and oscillations of various types, shapes and sizes. They are very useful in cosmology and quantum field theory in curved backgrounds. Further, we found the application of ultraspherical polynomials in vibrational calculations in the structure of the $Ca^{+}_{-} H_2$ exciplex, in the state correlated with 3D calcium ion state. For this ultraspherical polynomials are used for formation of a basis set for a bending mode. Ultraspherical polynomial expansions are used to mitigate Gibb's phenomenon Fourier-Bessel series solutions of a dynamic sphere problem. Also, these orthogonal polynomials are of great importance in approximation theory, physics and the mathematical theory of mechanical quadratures etc.

The aim of present paper is to derive the generating functions for the ultraspherical polynomials by using the Truesdell's method, giving suitable interpretation to the index *n*. It is worth recalling that this method yields two generating functions for the ultraspherical polynomials, independently from ascending and descending recurrence relations, where as the simultaneous use of these recurrence relations in other group theoretic methods. The results obtained for ultraspherical polynomials are well known in the theory of special functions[8,9].

Definition

The ultraspherical polynomials[10] can be defined interms of the recurrence relations :

$$C_0^{\upsilon}(x) = 1,$$

$$C_1^{\upsilon}(x) = 2\upsilon x$$

and $$C_n^{\upsilon}(x) = \frac{1}{2}\left[2x\,(n+\upsilon-1)\,C_{n-1}^{\upsilon}(x) - (n+2\upsilon-2)\,C_{n-2}^{\upsilon}(x)\right] \qquad \ldots(1)$$

In order to obtain generating functions for the set, we begin with two in independent (descending and ascending) recurrence relations satisfied by each element of this set:

$$D\,C_n^{\upsilon}(x) = \frac{1}{(1-x^2)}\left[(2\upsilon-1+n)\,C_{n-1}^{\upsilon}(x) - n\,x\,C_n^{\upsilon}(x)\right] \qquad \ldots(2)$$

and

$$D\,C_n^{\upsilon}(x) = \frac{1}{(1-x^2)}\left[-(n+1)\,C_{n+1}^{\upsilon}(x) + (2\upsilon+n)x\;C_n^{\upsilon}(x)\right] \qquad \ldots(3)$$

These two independent differential recurrence relations determine the linear ordinary differential equation –

$$(1-x^2)D^2C_n^{\upsilon}(x) - (2\upsilon+1)xDC_n^{\upsilon}(x) + n(2\upsilon+n)C_n^{\upsilon}(x) = 0$$

where $$D \equiv \frac{d}{dx} \qquad \ldots(4)$$

The proofs of these results are obvious.

Moreover, the following is the representation of ultraspherical polynomials in terms of Jacobi polynomials[7]:

$$C_n^{\upsilon}(x) = \frac{(2\upsilon)_n}{\left(\upsilon+\frac{1}{2}\right)_n}\,P_n^{\left(\upsilon-\frac{1}{2},\;\upsilon-\frac{1}{2}\right)}(x)$$

Generating function derived from the ascending recurrence relation

We shall use the Truesdell's F-equation[11a]

$$\frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1) \qquad \ldots(5)$$

to find a generating function for the set of polynomials $C_{\alpha+n}^{\upsilon}(x)$ as follows :

The polynomial $C_n^{\upsilon}(x)$ satisfies the ascending recurrence relation –

$$\frac{d}{dx}\left[C_n^{\upsilon}(x)\right]=\frac{1}{(1-x^2)}\left[-(n+1)C_{n+1}^{\upsilon}(x)+(2\upsilon+n)x\,C_n^{\upsilon}(x)\right] \qquad \ldots(6)$$

Let $f(y,\alpha)=C_n^{\upsilon}(x)$, so that we have

$$\frac{\partial}{\partial y}f(y,\alpha)=\frac{1}{1-y^2}[-(\alpha+1)f(y,\alpha+1)+(2\upsilon+\alpha)y\,f(y,\alpha)] \qquad \ldots(7)$$

This equation is called the f-type equation and can be written as –

$$\frac{\partial}{\partial y}f(y,\alpha)=A(y,\alpha)f(y,\alpha)+B(y,\alpha)f(y,\alpha+1)$$

With $A(y,\alpha)=\dfrac{(2\upsilon+\alpha)y}{1-y^2}$ and $B(y,\alpha)=\dfrac{-(\alpha+1)}{1-y^2}$

We shall transform $f(y,\alpha)$ into $g(y,\alpha)$ so that –

$$\frac{\partial}{\partial y}g(y,\alpha)=C(y,\alpha)\;g(y,\alpha+1)$$

Let

$$\begin{aligned} g(y,\alpha) &= \exp\left\{-\int_{y_0}^{y}A(\theta,\alpha)\,d\theta\right\}\; f(y,\alpha) \\ &= f(y,\alpha)\,\exp\left\{\int_{y_0}^{y}\frac{-(2\upsilon+\alpha)\theta}{1-\theta^2}\,d\theta\right\} \\ &= f(y,\alpha)(y^2-1)^{\upsilon+\frac{\alpha}{2}}(y_0^2-1)^{-\left(\upsilon+\frac{\alpha}{2}\right)} \end{aligned}$$

Now, if we write $y_0=\mu$, then we get

$$g(y,\alpha)=(\mu^2-1)^{-\left(\upsilon+\frac{\alpha}{2}\right)}(y^2-1)^{\upsilon+\frac{\alpha}{2}}f(y,\alpha) \qquad \ldots(8)$$

It can be easily verified that this satisfies g-type equation

$$\frac{\partial}{\partial y} g(y,\alpha) = (\mu^2 - 1)^{-\upsilon - \frac{\alpha}{2}} (\alpha + 1)(y^2 - 1)^{\upsilon + \frac{\alpha}{2} - 1} f(y, \alpha + 1) \qquad \ldots(9)$$

Let $C(y,\alpha)$ denote the factorable coefficient of $g(y,\alpha+1)$ in (9), then

$$C(y,\alpha) = (\alpha + 1)(\mu^2 - 1)^{\frac{1}{2}} (y^2 - 1)^{-\frac{3}{2}}$$

with $C(y,\alpha) = A(\alpha) Y(y)$,

where $A(\alpha) = (\alpha + 1)(\mu^2 - 1)^{\frac{1}{2}}$ and $Y(y) = (y^2 - 1)^{-\frac{3}{2}}$

We effect the transformation of $g(y,\alpha)$ into $F(z,\alpha)$ by letting

$$z = \int_{y_1}^{y} y(u)\, du = \frac{-y}{\sqrt{y^2 - 1}} + \frac{y_1}{\sqrt{y_1{}^2 - 1}}$$

and $F_0 F(z,\alpha) = g(y,\alpha) \exp\left\{ \int_{\alpha_0}^{\alpha} \log A(x) \Delta x \right\}$

On choosing $\alpha_0 = -1$, we get

$$\int_{-1}^{\alpha} \log\left\{ (x+1)(\mu^2 - 1)^{\frac{1}{2}} \right\} \Delta x = \int_{-1}^{\alpha} \log\,(x+1)\,\Delta x + \int_{-1}^{\alpha} \log\,(\mu^2 - 1)^{\frac{1}{2}}\,\Delta x$$

Since $\int_{0}^{x} \log z \Delta z = \log \Gamma(x) - \log \sqrt{2\pi}$

$$= \log\left[\frac{\Gamma(\alpha + 1)(\mu^2 - 1)^{\frac{1}{2}(\alpha+1)}}{\sqrt{2\pi}} \right]$$

This implies

$$F_0 F(z,\alpha) = g(y,\alpha) \exp\left\{ \log\left[\frac{\Gamma(\alpha+1)\,(\mu^2 - 1)^{\frac{1}{2}(\alpha+1)}}{\sqrt{2\pi}} \right] \right\}$$

Further, if we choose $y_1 = 0$, $\alpha_0 = -1$ and $F_0 = \frac{1}{\sqrt{2\pi}}$

we get
$$F(z,\alpha) = \Gamma(\alpha+1)\ (\mu^2-1)^{\frac{1}{2}(\alpha+1)} g\left(\frac{-z}{\sqrt{z^2-1}},\alpha\right)$$

To show that $F(z,\alpha)$ does indeed satisfy the F-equation we determine $\frac{\partial}{\partial z}F(z,\alpha)$ as follows :

$$\frac{\partial}{\partial z}F(z,\alpha) = \Gamma(\alpha+1)(\mu^2-1)^{\frac{1}{2}(\alpha+1)}\frac{\partial}{\partial z}\left[g\left(\frac{-z}{\sqrt{z^2-1}},\alpha\right)\right]$$

$$= \Gamma(\alpha+1)(\alpha+1)(\mu^2-1)^{\frac{1}{2}}(\mu^2-1)^{\frac{1}{2}(\alpha+1)}(z^2-1)^{-3/2}\left[\frac{z^2}{z^2-1}-1\right]^{-3/2} g\left(\frac{-z}{\sqrt{z^2-1}},\alpha-1\right)$$

$$= \Gamma(\overline{\alpha+1}+1)\,(\mu^2-1)^{\frac{1}{2}(\overline{\alpha+1}+1)} g\left(\frac{-z}{\sqrt{z^2-1}},\alpha+1\right)$$

$$= F(z,\alpha+1)$$

Thus, we have –

$$\frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1)$$

We express $F(z,\alpha)$ in the following form –

$$F(z,\alpha) = \Gamma(\alpha+1)(\mu^2-1)^{\frac{1}{2}(\alpha+1)} g\left(\frac{-z}{\sqrt{z^2-1}},\alpha\right)$$

$$= \Gamma(\alpha+1)(\mu^2-1)^{-\upsilon+\frac{1}{2}}(z^2-1)^{-\upsilon-\frac{\alpha}{2}} f\left(\frac{-z}{\sqrt{z^2-1}},\alpha\right)$$

$$= \Gamma(\alpha+1)\,(\mu^2-1)^{-\upsilon-\frac{1}{2}}\left(z^2-1\right)^{-\upsilon-\frac{\alpha}{2}} C_\alpha^\upsilon\left[\frac{-z}{\sqrt{z^2-1}}\right]$$

Since $F(z+y,\alpha)=\Gamma(\alpha+1)\left(\mu^2-1\right)^{-\nu-\frac{1}{2}}C_{\alpha}^{\upsilon}\left[\frac{-(z+y)}{\sqrt{(z+y)^2-1}}\right]$

and $F(z,\alpha+y)=\Gamma(\alpha+n+1)\left(\mu^2-1\right)^{-\upsilon+\frac{1}{2}}\left(z^2-1\right)^{-\upsilon-\frac{1}{2}(\alpha+n)}C_{\alpha+n}^{\upsilon}\left[\frac{-z}{\sqrt{z^2-1}}\right]$

From Truesdell's F-equation generating function theorem[11b], we get –

$$F(z+y,\alpha)=\sum_{n=0}^{\infty}\frac{y^n}{n!}F(z,\alpha+n) \qquad \ldots(10)$$

It follows that

$$\Gamma(\alpha+1)\left[(z+y)^2-1\right]^{-\upsilon-\frac{\alpha}{2}}C_{\alpha}^{\upsilon}\left[\frac{-(z+y)}{\sqrt{(z+y)^2-1}}\right]$$

$$=\sum_{n=0}^{\infty}\frac{y^n}{n!}\Gamma(\alpha+n+1)(z^2-1)^{-\upsilon-\frac{\alpha}{2}-\frac{n}{2}}C_{\alpha+n}^{\upsilon}\left[\frac{-z}{\sqrt{z^2-1}}\right]$$

or

$$\left[1+\left(\frac{y^2+2zy}{z^2-1}\right)\right]^{-\upsilon-\frac{\alpha}{2}}C_{\alpha}^{\upsilon}\left[\frac{-z-y}{\sqrt{z^2+y^2+2zy-1}}\right]$$

$$=\sum_{n=0}^{\infty}\binom{\alpha+n}{\alpha}(z^2-1)^{-\frac{n}{2}}C_{\alpha+n}^{\upsilon}\left[\frac{-z}{\sqrt{z^2-1}}\right]y^n$$

Replacing $y(z^2-1)^{-\frac{1}{2}}$ by t and $-z\,(z^2-1)^{-\frac{1}{2}}$ by *x*, we get –

$$(1-2xt+t^2)^{-\upsilon-\frac{\alpha}{2}}C_{\alpha}^{\upsilon}\left[\frac{x-t}{\sqrt{1-2xt+t^2}}\right]=\sum_{n=0}^{\infty}\binom{\alpha+n}{\alpha}C_{\alpha+n}^{\upsilon}(x)(t)^n \qquad \ldots(11)$$

which is a well-known generating relation for $C_n^{\upsilon}(x)$

Generating function derived from the Descending Recurrence Relation

In a similar manner, we shall use the Truesdell's G-equation[11c]

$$\frac{\partial}{\partial z} G(z,\alpha) = G(z,\alpha-1) \qquad \ldots(12)$$

to derive a generating relation for the set of functions $C^{\upsilon}_{\alpha-n}(x)$ as follows:

The Polynomial $C^{\upsilon}_{n}(x)$ satisfies the descending recurrence relation

$$\frac{d}{dx} C^{\upsilon}_{n}(x) = \frac{1}{1-x^2}\left[(2\upsilon-1+n)\,C^{\upsilon}_{n-1}(x) - nx\,C^{\upsilon}_{n}(x)\right] \qquad \ldots(13)$$

Let $f(y,\alpha) = C^{\upsilon}_{\alpha}(x)$, so that we have

$$\frac{\partial}{\partial y} f(y,\alpha) = \frac{1}{1-y^2}[(2\upsilon-1+\alpha)\,f(y,\alpha-1) - \alpha y\, f(y,\alpha)] \qquad \ldots(14)$$

which can be written as –

$$\frac{\partial}{\partial y} f(y,\alpha) = A(y,\alpha)\, f(y,\alpha) + B(y,\alpha)\, f(y,\alpha-1)$$

with $\quad A(y,\alpha) = \frac{-\alpha y}{1-y^2},\ B(y,\alpha) = \frac{2\upsilon-1+\alpha}{1-y^2}$

Now, we shall transform $f(y,\alpha)$ into $g(y,\alpha)$

so that $\frac{\partial}{\partial y} g(y,\alpha) = C(y,\alpha)\, g(y,\alpha-1)$

Therefore, let us suppose that –

$$g(y,\alpha) = f(y,\alpha)\ \exp\left\{-\int_{y_0}^{y} A(\theta)\, d\theta\right\}$$

$$= (1-\mu^2)^{\frac{\alpha}{2}}(1-y^2)^{-\frac{\alpha}{2}}\, f(y,\alpha)$$

On choosing $y_0 = \mu$

Thus $\quad \frac{\partial}{\partial y} g(y,\alpha) = (2\upsilon-1+\alpha)\ (1-y^2)^{-\frac{3}{2}}(1-\mu^2)^{\frac{1}{2}}\ \ g(y,\alpha-1) \qquad \ldots(15)$

Let $C(y,\alpha)$ denote the factorable notation of $g(y, \alpha-1)$

then $C(y,\alpha)= (2\upsilon-1+\alpha)\,(1-y^2)^{-\frac{3}{2}}(1-\mu^2)^{\frac{1}{2}}$

with $Y(y)=(1-y^2)^{-\frac{3}{2}}$ and $A(\alpha)=(2\upsilon-1+\alpha)(1-\mu^2)^{\frac{1}{2}}$

We effect the transform of $g(y,\alpha)$ in $G(z,\alpha)$ by taking –

$$z=\int_{y_1}^{y} Y(y)dy$$

$$=\frac{y}{\sqrt{1-y^2}}-\frac{y_1}{\sqrt{1-y_1{}^2}}$$

On choosing $y_1 = 0$, it becomes –

$$z=\frac{y}{\sqrt{1-y^2}} \quad \text{or} \quad y=\frac{z}{\sqrt{z^2+1}}$$

Now $$h(\alpha)=\exp\left\{-\int_{\alpha_0}^{\alpha+1}\log A(\beta)\,d\beta\right\}$$

If we choose $\alpha_0=0$, we get

$$=\exp\left\{-\int_{0}^{\alpha+1}\log[(2\upsilon-1+\beta\}(1-\mu^2)^{\frac{1}{2}}]d\beta\right\}$$

$$=(1-\mu^2)^{-\frac{1}{2}(\alpha+1)}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}$$

We have $G_0 G(z,\alpha)=h(\alpha)g(y,\alpha)$. On choosing $G_0=1$, it gives that

$$G(z,\alpha)=(1-\mu^2)^{-\frac{1}{2}(\alpha+1)}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}g\left(\frac{z}{\sqrt{z^2+1}},\alpha\right)$$

Finally, on choosing $y_1=0$, $\alpha_0=0$ and $G_0=1$ we have

$$G(z,\alpha) = (1-\mu^2)^{-\frac{1}{2}(\alpha+1)}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}\,g\left[\frac{z}{\sqrt{z^2-1}},\alpha\right]$$

To Show that $G(z,\alpha)$ does indeed satisfy the G-equation we determine $\frac{\partial}{\partial z}G(z,\alpha)$ as follows:

$$\frac{\partial}{\partial z}G(z,\alpha) = (1-\mu^2)^{-\frac{1}{2}(\alpha+1)}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}\frac{\partial}{\partial z}g\left[\frac{z}{\sqrt{z^2+1}},\alpha\right]$$

$$= (1-\mu^2)^{-\frac{\alpha}{2}}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon-1)}g\left[\frac{z}{\sqrt{z^2+1}},\alpha-1\right]$$

$$= G(z,\alpha-1)$$

It can be easily seen that

$$G(z,\alpha)=(1-\mu^2)^{-\frac{\alpha}{2}}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}(z^2+1)^{\frac{\alpha}{2}}\,C_\alpha^\upsilon\left[\frac{z}{\sqrt{z^2+1}}\right]$$

From Truesdell's G-equation generating function theorem[11d], we have

$$G(z+y,\,\alpha)=\sum_{n=0}^{\infty}\frac{y^n}{n!}G(z,\alpha-n) \qquad \ldots(16)$$

which implies that

$$G(z+y,\,\alpha) = (1-\mu^2)^{-\frac{1}{2}}\,\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon)}\left[(z+y)^2+1\right]^{\frac{\alpha}{2}}\,C_\alpha^\upsilon\left[\frac{z+y}{\sqrt{(z+y)^2+1}}\right]$$

and

$$G(z,\alpha-n)=(1-\mu^2)^{-\frac{1}{2}}\frac{\Gamma(2\upsilon-1)}{\Gamma(\alpha+2\upsilon-n)}(z^2-1)^{\frac{1}{2}(\alpha-n)}C_{\alpha-n}^\upsilon\left[\frac{z}{\sqrt{z^2+1}}\right]$$

From the theorem, we have –

$$\left[(z+y)^2+1\right]^{\frac{\alpha}{2}}C_\alpha^\upsilon\left[\frac{z+y}{\sqrt{(z+y)^2+1}}\right]$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{\Gamma(\alpha+2\upsilon)}{\Gamma(\alpha+2\upsilon-n)} (z^2-1)^{\frac{1}{2}(\alpha-n)} C_{\alpha-n}^{\upsilon}\left[\frac{z}{\sqrt{z^2+1}}\right].$$

Replacing $\frac{z}{\sqrt{z^2+1}}$ by x and $\frac{y}{\sqrt{z^2+1}}$ by $-t$, we get –

$$(1-2xt+t^2)^{\frac{\alpha}{2}} C_{\alpha}^{\upsilon}\left[\frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}}\right] = \sum_{n=0}^{\infty} \frac{(1-\alpha-2\upsilon)_n}{n!} C_{\alpha-n}^{\upsilon}(x)\, t^n \qquad \ldots(17)$$

which is the well-known generating relation for $C_n^{\upsilon}(x)$.

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