

## D-Radius and D-Diameter of Some Families of Graphs

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### Abstract

The D-distance between vertices of a graph is obtained by considering the path lengths and as well as the degrees of vertices present on the path. In this article, we study the D-radius and D-diameter of some families of graphs with respect to D-distance. 2000 Mathematics Subject Classification: 05C12.

**Keywords:** D-distance; D-radius; D-diameter

### Introduction

By a graph  $G=(V, E)$ , we mean a finite undirected graph without loops and multiple edges. The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, Hamilton city problems, extremal problems on connectivity and diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs [1].

In an earlier article authors introduced the concept of D-distance [2], by considering not only path length between vertices, but also the degrees of all vertices present in a path while defining the D-distance.

### Preliminaries

Throughout this article, by a graph  $G(V, E)$  or simply  $G$ , we mean a non-trivial, finite, undirected graph without multiple edges and loops. Further all graphs we consider are connected.

**Definition 2.1:** If  $u, v$  are vertices of a connected graph  $G$  the D-length

$P$  of a  $u$ - $v$  path  $S$  is defined as  $l(s)=d(u, v)+\deg(u)+\deg(v)+\sum \deg(w)$  where sum runs over all intermediate vertices  $w$  of  $s$ .

**Definition 2.2:** (D-distance). The  $D$ -distance  $d^D(u, v)$  between two vertices  $u, v$  of a connected graph  $G$  is defined

as  $d^D(u, v) = \min\{l(s)\}$  if  $u, v$  are distinct and  $d^D(u, v) = 0$  if  $u = v$ , where the minimum is taken over all  $u - v$  paths  $s$  in  $G$ .

**Definition 2.3:** The  $D$ -eccentricity of any vertex  $v$ ,  $e^D(v)$ , is defined as the maximum distance from  $v$  to any other vertex, i.e.,  $e^D(v) = \max\{d^D(u, v) : u \in V(G)\}$

**Definition 2.4:** Any vertex  $u$  for which  $d^D(u, v) = e^D(v)$  is called  $D$  eccentric vertex of  $v$ . Further, a vertex  $u$  is said to be  $D$  eccentric vertex of  $G$  if it is the  $D$ -eccentric vertex of some vertex.

**Definition 2.5:** The  $D$ -radius, denoted by  $r^D(G)$ , is the minimum  $D$ -eccentricity among all vertices of  $G$  i.e.,  $r^D(G) = \min\{e^D(v) : v \in V(G)\}$ . Similarly the  $D$ -diameter,  $d^D(G)$ , is the maximum  $D$ -eccentricity among all vertices of  $G$ .

**Definition 2.6:** The  $D$ -center of  $G$ ,  $C^D(G)$ , is the sub graph induced by the set of all vertices of minimum  $D$ -eccentricity. A graph is called  $D$ -self-centered if  $C^D(G) = G$  or equivalently  $r^D(G) = d^D(G)$ . Similarly, the set of all vertices of maximum  $D$ -eccentricity is the  $D$ -periphery of  $G$ .

### D-radius and D-diameter of families of graphs

There are some graphs for which D-radius and D-diameter are same, i.e., they are D-self-centered.

**Theorem 3.1:** For Complete graph,  $K_n$ , on  $n$  vertices ( $n \geq 3$ ) we have  $r^D(K_n) = d^D(K_n) = 2n - 1$ .

**Proof:** In complete graph  $K_n$ , degree of each vertex is  $n - 1$ . Thus D-distance between any two vertices is  $1 + n - 1 + n - 1 = 2n - 1$ . Thus D-eccentricity of every vertex is  $2n - 1$ . Hence D-radius and D-diameter of  $K_n$  are equal and is equal to  $2n - 1$ .

**Theorem 3.2:** For cycle graph,  $C_n$ , with  $n$  vertices we have  $r^D(C_n) = d^D(C_n) = \begin{cases} \frac{3n+4}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

**Proof:** In cycle graph  $C_n$  degree of each vertex is  $2$ . The D-distance between the vertices is given below. We will consider even and odd cases separately.

**Case 1:** When  $n$  is even, the D-distance in  $C_n$  are as below.

TABLE 1. D-distances of cyclic graphs if n is even.

	$v_1$	$v_2$	$v_3$	...	$v_{\frac{n}{2}-1}$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	$v_{\frac{n}{2}+3}$	...	$v_{n-1}$	$v_n$
$v_1$	0	5	8	...	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	...	8	5
$v_2$	5	0	5	...	$\frac{3n-14}{2}$	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	...	11	8
$v_3$	8	5	0	...	$\frac{3n-20}{2}$	$\frac{3n-14}{2}$	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	...	14	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n}{2}-1}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	$\frac{3n-20}{2}$	...	0	5	8	11	14	...	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$
$v_{\frac{n}{2}}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	...	5	0	5	8	11	...	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$
$v_{\frac{n}{2}+1}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	...	8	5	0	5	8	...	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$
$v_{\frac{n}{2}+2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	...	11	8	5	0	5	...	$\frac{3n-14}{2}$	$\frac{3n-8}{2}$
$v_{\frac{n}{2}+3}$	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	...	14	11	8	5	0	...	$\frac{3n-20}{2}$	$\frac{3n-14}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{n-1}$	8	11	14	...	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	$\frac{3n-20}{2}$	...	0	5
$v_n$	5	8	11	...	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	...	5	0

From the above table, we see that the D-eccentricity of every vertex is  $\frac{3n+4}{2}$ . Hence the D- radius and D-diameter is same, if  $n$  is even.

**Case 2:** When  $n$  is odd, the D-distance in  $C_n$  are as below

**TABLE 2. D-distances of cyclic graphs if n is odd.**

	$v_1$	$v_2$	$v_3$	...	$\frac{v_{n-1}}{2}$	$\frac{v_{n+1}}{2}$	$\frac{v_{n+3}}{2}$	$\frac{v_{n+5}}{2}$	...	$v_{n-1}$	$v_n$
$v_1$	0	5	8	...	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	...	8	5
$v_2$	5	0	5	...	$\frac{3n-11}{2}$	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$	...	11	5
$v_3$	8	5	0	...	$\frac{3n-17}{2}$	$\frac{3n-11}{2}$	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$	...	14	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{v_{n-1}}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	$\frac{3n-17}{2}$	...	0	5	8	11	...	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$
$\frac{v_{n+1}}{2}$	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	...	5	0	5	8	...	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$
$\frac{v_{n+3}}{2}$	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	...	8	5	0	5	...	$\frac{3n-11}{2}$	$\frac{3n-5}{2}$
$\frac{v_{n+5}}{2}$	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$	...	11	8	5	0	...	$\frac{3n-17}{2}$	$\frac{3n-11}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{n-1}$	8	11	14	...	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	$\frac{3n-17}{2}$	...	0	5
$v_n$	5	8	11	...	$\frac{3n+1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	...	5	0

From the above table, we can see that the D-eccentricities of each vertex is  $\frac{3n+1}{2}$ , if  $n$  is odd. Hence D-radius and D-

diameter is same. Thus  $r^D(C_n) = d^D(C_n) = \begin{cases} \frac{3n+4}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$ .

Next, we calculate D-radius and D-diameter of some families of graphs.

**Theorem 3.3:** For complete biaparted graph  $K_{m,n}, (m \geq n)$  we have  $r^D(K_{m,n}) = m + 2(n + 1)$  and  $d^D(K_{m,n}) = n + 2(m + 1)$ .

**Proof:** Without loss of generality assume that  $(m \geq n)$ .  $N$  complete biparted graph  $K_{m,n}$ , degree of the vertices  $v_1, v_2, v_3, \dots, v_m$  is  $n$  and degree  $u_1, u_2, u_3, \dots, u_n$  of the vertices  $m$ . Thus  $d^D(v_i, v_j) = m + 2(n + 1)$  if  $i \neq j$ ,  $d^D(u_i, u_j) = n + 2(m + 1)$  if  $i \neq j$  and  $d^D(v_i, u_j) = m + n + 1$ .  
Therefore,  $e^D(v_i) = m + 2(n + 1) \forall i$  and  $e^D(u_j) = n + 2(m + 1) \forall j$ . Hence the D-radius of  $K_{m,n}$  is  $m + 2(n + 1)$  and D-diameter of  $K_{m,n}$  is  $n + 2(m + 1)$ .

**Theorem 3.4:** For star graph,  $st_{n,1}$ , we have  $r^D(st_{n,1}) = n + 2$  and  $d^D(st_{n,1}) = n + 4$ .

**Proof:** In star graph,  $st_{n,1}$ , the degree of central vertex is  $n$  and degree of remaining vertices is 1. The D-distance from central vertex to other vertices is  $n + 2$  and the D-distance between all other vertices is  $n + 4$ . Therefore the minimum D-eccentricity is  $n + 2$  and the maximum D-eccentricity is  $n + 4$ . Hence the D-radius of  $st_{n,1}$  is  $n + 2$  and the D-diameter of  $st_{n,1}$  is  $n + 4$ .

**Theorem 3.5:** For path graph  $P_n$ , on  $n$  vertices  $(n \geq 3)$  we have  $r^D(P_n) = \begin{cases} \frac{3n-1}{2} & \text{if } n \text{ is odd} \\ \frac{3n+2}{2} & \text{if } n \text{ is even} \end{cases}$  and

$$d^D(P_n) = 3(n-1) \text{ for } n \geq 3.$$

Further  $r^D(P_2) = d^D(P_2) = 3$ .

**Proof:** In path graph  $P_n$ , degree of end vertices is 1 and remaining vertices of degree is 2.

For  $n = 2$ ,  $d^D(v_1, v_2) = 3$  and  $d^D(v_1, v_1) = d^D(v_2, v_2) = 0$ . Thus the D-eccentricity of each vertex is 3. Hence the D-radius and D-diameter of  $P_2$  are same and equal to 3.

Next consider path graph with  $n \geq 3$ . we can consider even and odd cases separately, in this case the D-distance between vertices are as given below.

**Case 1:** When  $n$  is odd, the D-distance in  $P_n$  are as below

**TABLE 3. D-distances of path graphs if n is odd.**

	$v_1$	$v_2$	$v_3$	...	$\frac{v_{n-1}}{2}$	$\frac{v_{n+1}}{2}$	$\frac{v_{n+3}}{2}$	...	$v_{n-1}$	$v_n$
$v_1$	0	4	7	...	$\frac{3n-7}{2}$	$\frac{3n-1}{2}$	$\frac{3n+5}{2}$	...	$3n-5$	$3n-3$
$v_2$	4	0	5	...	$\frac{3n-11}{2}$	$\frac{3n-5}{2}$	$\frac{3n+1}{2}$	...	$3n-7$	$3n-5$
$v_3$	7	5	0	...	$\frac{3n-17}{2}$	$\frac{3n-11}{2}$	$\frac{3n-5}{2}$	...	$3n-10$	$3n-7$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\frac{v_{n-1}}{2}$	$\frac{3n-7}{2}$	$\frac{3n-11}{2}$	$\frac{3n-17}{2}$	...	0	5	8	...	$\frac{3n+1}{2}$	$\frac{3n+5}{2}$
$\frac{v_{n+1}}{2}$	$\frac{3n-1}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	...	5	0	5	...	$\frac{3n-5}{2}$	$\frac{3n-1}{2}$
$\frac{v_{n+3}}{2}$	$\frac{3n+5}{2}$	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	...	8	5	0	...	$\frac{3n-11}{2}$	$\frac{3n-7}{2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$v_{n-1}$	$3n-5$	$3n-7$	$3n-10$	...	$\frac{3n+1}{2}$	$\frac{3n-5}{2}$	$\frac{3n-11}{2}$	...	0	4
$v_n$	$3n-3$	$3n-5$	$3n-7$	...	$\frac{3n+5}{2}$	$\frac{3n-1}{2}$	$\frac{3n-7}{2}$	...	4	0

From the above table, we can see that the D-eccentricities are

$\left\{ 3n-3, 3n-5, 3n-7, \dots, \frac{3n+5}{2}, \frac{3n-1}{2}, \frac{3n+5}{2}, \dots, 3n-5, 3n-3 \right\}$ . Thus the maximum D-eccentricity of  $P_n$  is

$3n-3$  and minimum D-eccentricity is  $\frac{3n-1}{2}$ . Hence the D-radius is  $\frac{3n-1}{2}$  and the D-diameter is  $3n-3$  if  $n$  is odd.

**Case 2:** When  $n$  is even, the D-distance in  $P_n$  are as below

TABLE 4. D-distances of cyclic graphs if n is even.

	$v_1$	$v_2$	$v_3$	...	$v_{\frac{n-1}{2}}$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	...	$v_{n-1}$	$v_n$
$v_1$	0	4	7	...	$\frac{3n-10}{2}$	$\frac{3n-4}{2}$	$\frac{3n+2}{2}$	$\frac{3n+8}{2}$	...	$3n-5$	$3n-3$
$v_2$	4	0	5	...	$\frac{3n-14}{2}$	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	$\frac{3n+4}{2}$	...	$3n-7$	$3n-5$
$v_3$	7	5	0	...	$\frac{3n-20}{2}$	$\frac{3n-14}{2}$	$\frac{3n-8}{2}$	$\frac{3n-2}{2}$	...	$3n-10$	$3n-7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n-1}{2}}$	$\frac{3n-10}{2}$	$\frac{3n-14}{2}$	$\frac{3n-20}{2}$	...	0	5	8	11	...	$\frac{3n+4}{2}$	$\frac{3n+8}{2}$
$v_{\frac{n}{2}}$	$\frac{3n-4}{2}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	...	5	0	5	8	...	$\frac{3n-2}{2}$	$\frac{3n+2}{2}$
$v_{\frac{n}{2}+1}$	$\frac{3n+2}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	...	8	5	0	5	...	$\frac{3n-8}{2}$	$\frac{3n-4}{2}$
$v_{\frac{n}{2}+2}$	$\frac{3n+8}{2}$	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	...	11	8	5	0	...	$\frac{3n-14}{2}$	$\frac{3n-10}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{n-1}$	$3n-5$	$3n-7$	$3n-10$	...	$\frac{3n+4}{2}$	$\frac{3n-2}{2}$	$\frac{3n-8}{2}$	$\frac{3n-14}{2}$	...	0	4
$v_n$	$3n-3$	$3n-5$	$3n-7$	...	$\frac{3n+8}{2}$	$\frac{3n+2}{2}$	$\frac{3n-4}{2}$	$\frac{3n-10}{2}$	...	4	0

From the above table, we can see that the D-eccentricities are  $\left\{ 3n-3, 3n-5, 3n-7, \dots, \frac{3n+8}{2}, \frac{3n+2}{2}, \frac{3n+2}{2}, \frac{3n+8}{2}, \dots, 3n-5, 3n-3 \right\}$ . Thus the maximum D-eccentricity of

$P_n$  is  $3n-3$  and minimum D-eccentricity is  $\frac{3n+2}{2}$ . Hence the D-radius is  $\frac{3n+2}{2}$  and the D-diameter is  $3n-3$  if  $n$

is even.

$$\text{Thus } r^D(P_n) = \begin{cases} \frac{3n-1}{2} & \text{if } n \text{ is odd} \\ \frac{3n+2}{2} & \text{if } n \text{ is even} \end{cases}$$

and  $d^D(P_n) = 3(n-1)$  for  $n \geq 3$ .

**Theorem 3.6:** For wheel graph,  $W_{n,1}$ , with  $n+1$  vertices we have  $r^D(W_{n,1}) = n+4$  and  $d^D(W_{n,1}) = n+8$  for  $n \geq 6$ .

Further we have  $r^D(W_{3,1}) = d^D(W_{3,1}) = 7, r^D(W_{4,1}) = 8, d^D(W_{4,1}) = 11, r^D(W_{5,1}) = 9$  and  $d^D(W_{5,1}) = 11$ .

**Proof:** For  $n=3, d^D(v_i, v_j) = 7$  if  $v_i \neq v_j$ . Thus the D-eccentricity of every vertex is 7. Hence the D-radius and D-diameter of  $W_{3,1}$  is 7 (and hence  $W_{3,1}$  is self-centered).

For  $n=4, d^D(v_i, v_j) = 8$  if  $v_i$  is central vertex and  $d^D(v_i, v_j) = 7$  or 11, if  $v_i$  is not central vertex. Thus the D-eccentricity central vertex is 8 and the D-eccentricity of other vertices is 11. hence the D-radius of  $W_{4,1}$  is 8 and D-diameter of  $W_{4,1}$  is 11.

For  $n=5, d^D(v_i, v_j) = 9$  if  $v_i$  is central vertex and  $d^D(v_i, v_j) = 7$  or 11, if  $v_i$  is not central vertex. Thus the D-eccentricity central vertex is 9 and the D-eccentricity of other vertices is 11. hence the D-radius of  $W_{5,1}$  is 9 and D-diameter of  $W_{5,1}$  is 11.

In wheel graph  $W_{n,1}$  degree of central vertex is  $n$  and degree of remaining vertices is 3. Thus D-eccentricity of central vertex is  $n+4$  and D-eccentricity of remaining vertices is  $n+8$  for  $n \geq 6$ . Thus the D-radius of  $W_{n,1}$  is  $n+4$  and D-diameter of  $W_{n,1}$  is  $n+8$  for  $n \geq 6$ .

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