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Contracting similarity fixed point of a class of self-affine fractal

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ABSTRACT

By constructing a kind of self-affine fractal, contracting similarity fixed point of this kind of fractal is studied. The paper introduces the basic concept of the self-affine fractal, and compares it with the self-similar fractal to show their connection and difference. Then the paper introduces the definition of the contracting similarity fixed point and shows the application and importance of it. Through the mathematical model depiction of self-affine fractal, combined with the newly-defined eigenfunction, the paper gives the contracting mapping function of contracting similarity fixed point. Based on the contracting mapping function, the paper shows the formula of the contracting similarity fixed point of the self-affine fractal. As an application, the paper finally presents two relevant examples.

KEYWORDS

Fractal; Self-affine fractal; Contracting similarity fixed point.

INTRODUCTION

At present, theory of fractal geometry is widely applied in many areas of research by many scholars. Fractal geometry not only provides a new angle of view for many research areas, and to a certain extent, reveals the laws and characteristics of the nature[1,2,3]. The successful study of fractal geometry is graphs of self-similar fractal geometry and the corresponding application. However, the situation of self-similar fractal geometry is uniform expansion or the same ratios of expansion and compression in all directions. So, when the object does not satisfy the conditions, self-similar fractal geometry can not be used in research[4,5,6].

Affine geometry can be traced back to Euler period, Snapper described affine geometry as: roughly speaking, the study of affine geometry is the study of Euclidean geometry when the length, area, angle and so on are removed as much as possible. People may think the affine geometry is a poor subject, on the contrary, it is quite rich. Theory of iterated function system is an efficient mathematical tool for the study of the properties of self-affine fractal. Iteration function system is defined on the basis of compression mapping, it is the most vibrant method for applying theory of fractal geometry to the graphics processing[7]. Iteration function system is a kind of creative method for the study of fractal geometric graphs. The system can carry out mathematical modeling of a complex graph through similarity transformation or affine transformation. The main contents of the system are compression mapping, metric spaces, as well as the measure theory and so on.

Self-similar fractal is an important research area of fractal geometry. In non-linear fractal, self-affine fractal is the closest one to linear fractal, so Mandelbrot originally classified them as the same group. Complex irregular phenomena of nature are mostly self-affine fractal rather than simple self-similar fractal, so self-affine fractal is very useful[8,9,10]. Self-affine fractal can better reflect the complexity and diversity of nature than self-similar fractal, so the content and expression form of the former are much more complex.

Self-affine set is one important set of fractal geometry, in which self-similar set is a special case. The difference between self-affine transformation and self-similar mapping is that self-affine transformation has different ratios of expansion and compression in different directions[11]. Affine transformation may be translation, rotation, expansion and compression and even the combination of the reflection.

Affine is non-uniform linear transformation, while similarity is a homogeneous linear transformation, which is a special case of the affine. Therefore, in the study of fractal geometry, the uniform linear fractal --- self-similar fractal is generally called linear fractal, and the rest are called nonlinear fractal[12,13].

Self-similar fractal is the basis of the study of nonlinear fractal, and the special case of nonlinear fractal is self-similar fractal. Strict linear fractal is strictly self-similar, that is there is an infinite nested mathematical structure[14,15,16]. In nonlinear fractal, the self-affine fractal is closest to linear fractal. Therefore, the research of nonlinear fractal are generally based on linear fractal, and the self-affine fractal is a breakthrough.

The paper mainly studies contracting similarity fixed point of self-affine fractal which is based on the Sierpinski gasket. By defining the self-affine fractal on R^2 and combing with the newly-defined eigenfunction, the paper gives the contracting mapping function of self-affine fractal[17]. With the contracting mapping function, the paper shows the formula of the contracting similarity fixed point of the self-affine fractal. As an application, the paper finally presents two relevant examples.

As the main result of the paper, we have

Theorem

If $i_1 i_2 \cdots i_n \in J_n$, the contracting similarity fixed point of a class of self-affine fractal can be described as the following formula:

$$(\frac{1}{2(1-r_{i_1}\cdots r_{i_n})}\sum_{k=1}^{n}(i_k-1)(1-r_{i_n})r_{i_1}\cdots r_{i_{k-1}},\ \frac{\sqrt{3}}{2(1-r_{i_1}\cdots r_{i_{k-1}})}\sum_{k=1}^{n}D(i_k)(1-r_{i_n})r_{i_1}\cdots r_{i_{k-1}}).$$

SOME DEFINITIONS

Definition[18]

IF $E \in R^n$. DENOTE $d(P_1, P_2)$ BY

$$|P_1 - P_2|, \forall P_1(x_1, y_1), P_1(x_1, y_1) \in E. \quad \textbf{(1)}$$

Definition[19]

Let $E \subset R^n$ be closed. A map $S_i : D \to D$ is called an affine compression mapping, if r_i ($0 < r_i < 1$) exists in the following formula such that

$$|S_i(x) - S_i(y)| = r_i |x - y|, \quad \forall x, y \in D. \quad \textbf{(2)}$$

Definition[20]

If $\{S_1, \cdots, S_n\}$ is an iterated function system (IFS), the set E satisfies the open set condition (OSC). If there is a bounded nonempty open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V. \quad \textbf{(3)}$$

Denote by J_n the set of all $n-$ sequences $\{i_1, i_2, \cdots, i_n\}$, where $1 \le i_1, i_2, \cdots, i_n \le 3, n \ge 1$.

Put

$$E_{i_1 i_2 \cdots i_n} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n} = x. \quad \textbf{(4)}$$

Coding system of copies at various levels of a set of a class of self-affine fractal are creatively presented in this paper. By using the newly-coded digital system, the paper can depict the changes of the three different directions of self-affine fractal copies at any levels. According to the newly-found contracting mapping function of self-affine fractal, combing with the related theorem and conclusion, the paper concludes the unified formula of contracting similarity fixed point of a class of self-affine fractal.

Definition[21]

Let $E \subset R^2$ satisfies OSC and $(x, y) \in E$, (x, y) is called a contracting similarity fixed point of E, if there are $n \in N^+$ and $(i_1, i_2, \cdots, i_n) \in J_n$ such that

$$S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(x, y) = (x, y). \quad \textbf{(5)}$$

Contracting similarity fixed point is one of the frontier research in the study of fractal geometry, and the study of it is of great significance for understanding the structure of self-similar set and promoting the development of fractal geometry[22].

If contracting similarity fixed point of self-similar set can be found, the local property of self-similar set can be obtained, which in turn the property of self-similar set can be got according to the dense everywhere of contracting similarity fixed point[23,24]. At present most scholars mainly focus on the study of the contracting similarity fixed point of self-similar fractal, while the study of the contracting similarity fixed point of self-affine fractal has not been found yet up to now.

THE STRUCTURE OF A CLASS OF SELF-AFFINE FRACTAL

We denote the three vertices of the class of the self-affine fractal as

$$\begin{cases} A=(0,0) \\ B=(1/2,\sqrt{3}/2)\,. \\ C=(1,0) \end{cases} \qquad \textbf{(6)}$$

and denote

$$E_0=\left\{(x,y):0\le y\le\sqrt{3}x,\ y\le-\sqrt{3}(x-1)\right\}.\textbf{(7)}$$

In the direction of the vertex A, copy progressively by using contracting ratio r_1. In the direction of the vertex B, copy progressively by using contracting ratio r_2. In the direction of the vertex C, copy progressively by using contracting ratio r_3.

Assumed that r_1, r_2, r_3 satisfy:

$$0<r_1+r_2,\, r_2+r_3,\, r_3+r_1\le 1\,. \qquad \textbf{(8)}$$

Assuming that self-affine contracting mapping of the three vertices are shown as below:

Definition

For each $(x,y)\in E_0$, let

$$\begin{cases} S_1(x,y)=(r_1x,r_1y); \\ S_2(x,y)=(r_2x+(1-r_2)/2, \\ \qquad r_2y+\sqrt{3}(1-r_2)/2) \\ S_3(x,y)=(r_3x+1-r_3,\, r_3y). \end{cases}; \qquad \textbf{(9)}$$

So,

$$E_n=\left\{S_{i_1}\circ\cdots\circ S_{i_n}(E_0):1\le i_1,\ \cdots,i_n\le 3\right\}. \qquad \textbf{(10)}$$

We obtain

$$E_0\supset E_1\supset\cdots\supset E_n\supset\cdots \qquad \textbf{(11)}$$

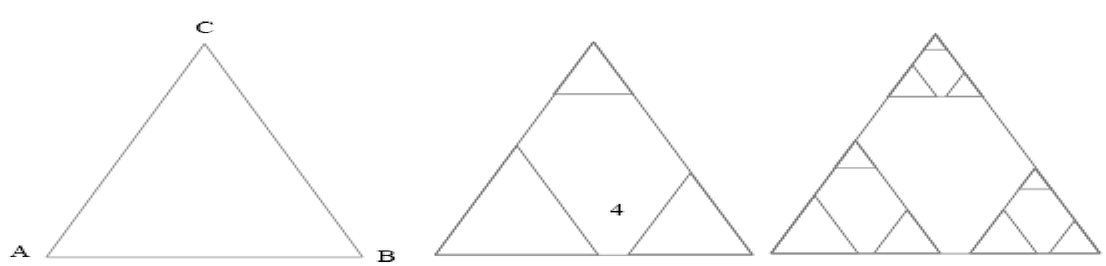

Figure 1: The self-affine fractal

The non-empty set $S=\cap_{n=0}^{\infty}E_n$ is the self-affine fractal (See Figure.1).

Remark

If $r_1=r_2=r_3$ and contracting ratios of the three vertices of self-affine fractal are the same, self-affine fractal will change into self-similar fractal.

SOME LEMMAS

Lemma

If $E\subset R^n$, E satisfies OSC, and F is an arbitrary n-copy of $E(k\geq 1)$, that is, there exists $(i_1,i_2,\cdots,i_n)\in J_n$ such that

$$F=S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_k}(E). \quad \textbf{(12)}$$

Then F must have a contracting similarity fixed point in E.

Based on the mathematical model of self-defined self-affine fractal, the paper shows the digital processing of self-affine fractal. With the model of self-affine fractal and the special newly-defined eigenfunction, together with the related techniques of fractal geometry, the paper gives the corresponding self-affine contracting mapping function of self-affine fractal. In order to prove the lemma 13, a new characteristic function is defined as below.

Definition

Let

$$D(i_k)=\begin{cases}1, & i_k=2;\\ 0, & else.\end{cases}. \quad \textbf{(13)}$$

Lemma

If $(i_1,i_2,\cdots,i_n)\in J_n$, then the general formula of the $n-$copy of a class of self-affine fractal is

$$S_{i_1}\circ\cdot S_{i_2}\circ\cdot\cdot\circ S_{i_n}(x,y)=(r_{i_1}\cdots r_{i_n}x+\frac{1}{2}\sum_{k=1}^{n}\quad(i_k-1)(1-r_{i_n})r_{i_1}\cdot r_{i_2}\cdots r_{i_{k-1}},$$
$$r_{i_1}\cdots r_{i_n}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{n}D(i_k)(1-r_{i_n})r_{i_1}\cdots r_{i_{k-1}}).$$

Proof

When $n=1$,

(a) If $i_1=1$,

$$S_{i_1}(x,y)=S_1(x,y)=(r_1x,r_1y)=(r_1x+\frac{1}{2}(1-1)(1-r_1),\ r_1y+\frac{\sqrt{3}}{2}D(1)(1-r_1))$$

$$=(r_1x+\frac{1}{2}\sum_{k=1}^{1}(1-1)(1-r_1),\ \ r_1y+\frac{\sqrt{3}}{2}\sum_{k=1}^{1}D(1)(1-r_1))$$

$$=(r_{i_1}x+\frac{1}{2}\sum_{k=1}^{1}(i_k-1)(1-r_{i_1}),\ \ r_{i_1}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{1}D(i_k)(1-r_{i_1}))\,.$$

(b) If $i_1 = 2$,

$$S_{i_1}(x, y) = S_2(x, y)$$

$$= (r_2 x + \frac{1-r_2}{2}, r_2 y + \frac{\sqrt{3}(1-r_2)}{2})$$

$$= (r_2 x + \frac{1}{2}(2-1)(1-r_2), r_2 y + \frac{\sqrt{3}}{2} D(2)(1-r_2))$$

$$= (r_2 x + \frac{1}{2}\sum_{k=1}^{1}(2-1)(1-r_2), \ r_2 y + \frac{\sqrt{3}}{2}\sum_{k=1}^{1} D(2)(1-r_2) r_{k-1})$$

$$= (r_{i_1} x + \frac{1}{2}\sum_{k=1}^{1}(i_k - 1)(1-r_{i_1}), \ r_{i_1} y + \frac{\sqrt{3}}{2}\sum_{k=1}^{1} D(i_k)(1-r_{i_1})) .$$

(c) If $i_1 = 3$,

$$S_{i_1}(x, y) = S_3(x, y) = (r_3 x + 1 - r_3, r_3 y)$$

$$= (r_3 x + \frac{1}{2}(3-1)(1-r_3), r_3 y + \frac{\sqrt{3}}{2} D(3)(1-r_3))$$

$$= (r_3 x + \frac{1}{2}\sum_{k=1}^{1}(3-1)(1-r_3), \ r_3 y + \frac{\sqrt{3}}{2}\sum_{k=1}^{1} D(3)(1-r_3))$$

$$= (r_{i_1} x + \frac{1}{2}\sum_{k=1}^{1}(i_k - 1)(1-r_{i_1}), \ r_{i_1} y + \frac{\sqrt{3}}{2}\sum_{k=1}^{1} D(i_k)(1-r_{i_1})) .$$

When $n = m$, suppose the lemma is follow, that is,

$$S_{i_1} \circ \cdots \circ S_{i_m}(x, y) = (r_{i_1} \cdots r_{i_m} x + \frac{1}{2}\sum_{k=1}^{m}(i_k - 1)\ (1-r_{i_k}) r_{i_1} \cdot r_{i_2} \cdots r_{i_{k-1}},$$

$$r_{i_1} \cdots r_{i_m} y + \frac{\sqrt{3}}{2}\sum_{k=1}^{m} D(i_k)(1-r_{i_k}) r_{i_1} \cdot r_{i_2} \cdots r_{i_{k-1}})$$

(a) If $i_{m+1} = 1$,

$$S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m} \circ S_{i_{m+1}}(x, y)$$

$$= S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m} \circ S_1(x, y) = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m}(r_1 x, r_1 y)$$

$$= (r_{i_1} \cdots r_{i_m} \cdot r_1 x + \frac{1}{2}\sum_{k=1}^{m}(i_k - 1)(1-r_{i_k}) r_{i_1} \cdots r_{i_{k-1}}, \ r_{i_1} \cdots r_{i_m} r_1 y + \frac{\sqrt{3}}{2}\sum_{k=1}^{m} D(i_k)(1-r_{i_k}) r_{i_1} \cdots r_{i_{k-1}})$$

$$=(r_{i_1}\cdots r_{i_m}\cdot r_1 x+\frac{1}{2}\sum_{k=1}^{m}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{1-r_1}{2}(1-1)r_{i_1}\cdots r_{i_m},$$

$$r_{i_1}\cdots r_{i_m}r_1 y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{\sqrt{3}(1-r_1)}{2}D(1)r_{i_1}\cdots r_{i_m})$$

$$=(r_{i_1}\cdots r_{i_{m+1}}x+\frac{1}{2}\sum_{k=1}^{m}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{1}{2}(i_{m+1}-1)(1-r_{i_{m+1}})r_{i_1}\cdots r_{i_m}r_{i_{m+1}},$$

$$r_{i_1}\cdots r_{i_{m+1}}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{\sqrt{3}}{2}D(i_{m+1})(1-r_{i_{m+1}})r_{i_1}\cdots r_{i_m}r_{i_{m+1}})$$

$$=(r_{i_1}\cdots r_{i_{m+1}}x+\frac{1}{2}\sum_{k=1}^{m+1}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}},\ r_{i_1}\cdots r_{i_{m+1}}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m+1}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}).$$

(b) If $i_{m+1}=2$,

$$S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_m}\circ S_{i_{m+1}}(x,y)=S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_m}\circ S_2(x,y)$$

$$=S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_m}(r_2x+\frac{1-r_2}{2},\ r_2y+\frac{\sqrt{3}(1-r_2)}{2})$$

$$=(r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_2x+\frac{1-r_2}{2}r_{i_1}\cdot r_{i_2}\cdots r_{i_m}+\frac{1}{2}\sum_{k=1}^{m}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}},$$

$$r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_2y+\frac{\sqrt{3}(1-r_2)}{2}r_{i_1}\cdot r_{i_2}\cdots r_{i_m}+\frac{\sqrt{3}}{2}\sum_{k=1}^{m}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}})$$

$$=(r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_2x+\frac{1}{2}\sum_{k=1}^{m}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{1}{2}(2-1)(1-r_2)r_{i_1}\cdot r_{i_2}\cdots r_{i_m},$$

$$r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_2y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{\sqrt{3}}{2}D(2)(1-r_2)r_{i_1}r_{i_2}\cdots r_{i_m})$$

$$=(r_{i_1}\cdots r_{i_{m+1}}x+\frac{1}{2}\sum_{k=1}^{m}(i_k-1)(1-r_{i_m})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{1}{2}(i_{m+1}-1)(1-r_{i_{m+1}})r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_{i_{m+1}},$$

$$r_{i_1}\cdots r_{i_{m+1}}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}\ +\frac{\sqrt{3}}{2}D(i_{m+1})(1-r_{i_{m+1}})r_{i_1}\cdot r_{i_2}\cdots r_{i_m}\cdot r_{i_{m+1}})$$

$$=(r_{i_1}\cdots r_{i_{m+1}}x+\frac{1}{2}\sum_{k=1}^{m+1}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}},\ r_{i_1}\cdots r_{i_{m+1}}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{m+1}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}}).$$

(c) If $i_{m+1}=3$,

$$S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m} \circ S_{i_{m+1}}(x, y)$$

$$= S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m} \circ S_1(x, y) = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m}(r_3 x + 1 - r_3, r_3 y)$$

$$= (r_{i_1} \cdots r_{i_m} r_3 x + (1 - r_3) r_{i_1} \cdots r_{i_m} + \frac{1}{2}\sum_{k=1}^{m}(i_k - 1)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}},$$

$$r_{i_1} \cdots r_{i_m} r_3 y + \frac{\sqrt{3}(1 - r_3)}{2}\sum_{k=1}^{m} D(i_k) r_{i_1} \cdots r_{i_{k-1}})$$

$$= (r_{i_1} r_{i_2} \cdots r_{i_m} r_3 x + \frac{1}{2}\sum_{k=1}^{m}(i_k - 1)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}} + \frac{1}{2}(3 - 1)(1 - r_3) r_{i_1} \cdot r_{i_2} \cdots r_{i_m},$$

$$r_{i_1} r_{i_2} \cdots r_{i_m} r_3 y + \frac{\sqrt{3}}{2}\sum_{k=1}^{m} D(i_k)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}} + \frac{\sqrt{3}}{2} D(3)(1 - r_3) r_{i_1} r_{i_2} \cdots r_{i_m})$$

$$= (r_{i_1} \cdots r_{i_{m+1}} x + \frac{1}{2}\sum_{k=1}^{m}(i_k - 1)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}} + \frac{1}{2}(i_{m+1} - 1)(1 - r_{i_{m+1}}) r_{i_1} r_{i_2} \cdots r_{i_m} r_{i_{m+1}},$$

$$r_{i_1} \cdots r_{i_{m+1}} y + \frac{\sqrt{3}}{2}\sum_{k=1}^{m} D(i_k)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}} + \frac{\sqrt{3}}{2} D(i_{m+1})(1 - r_{i_{m+1}}) r_{i_1} r_{i_2} \cdots r_{i_m} r_{i_{m+1}})$$

$$= (r_{i_1} \cdots r_{i_{m+1}} x + \frac{1}{2}\sum_{k=1}^{m+1}(i_k - 1)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}}, r_{i_1} \cdots r_{i_{m+1}} y + \frac{\sqrt{3}}{2}\sum_{k=1}^{m+1} D(i_k)(1 - r_{i_k}) r_{i_1} \cdots r_{i_{k-1}}).$$

Then, the lemma 3 is proved.

PROOF OF THE MAIN THEOREM

This article puts forward the contracting similarity fixed point of the self-affine fractal for the first time. The research period of contracting similarity fixed point in the field of fractal geometry is short, so the study of this issue is mainly concentrated on the self-similar fractal currently. This paper presents self-affine fractal which is more general than self-similar fractal as the research direction, so as to study the corresponding contracting similarity fixed point. Study of this direction has not been found yet in this area until now.

Proof of the Main Theorem:

As we all known, self-affine fractal $S = \cap_{n=0}^{\infty} E_n$ satisfies OSC. By the lemma 12, S must have a contracting similarity fixed point in E_0.

Then, by the definition 4, let

$$S_{i1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(x, y) = (x, y).$$

So, by Lemma 6, we have,

$$(r_{i_1}\cdots r_{i_n}x+\frac{1}{2}\sum_{k=1}^{n}(i_k-1)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}},\ r_{i_1}\cdots r_{i_n}y+\frac{\sqrt{3}}{2}\sum_{k=1}^{n}D(i_k)(1-r_{i_k})r_{i_1}\cdots r_{i_{k-1}})=(x,y).$$

That is,

$$(x,y)=(\frac{1}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\sum_{k=1}^{n}(i_k-1)(1-r_{i_k})\ r_{i_1}\cdot r_{i_2}\cdots r_{i_{k-1}},$$

$$\frac{\sqrt{3}}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\sum_{k=1}^{n}D(i_k)(1-r_{i_k})r_{i_1}r_{i_2}\cdots r_{i_{k-1}}).$$

Then, the Theorem 1.1 is proved.

Remark

If $r_1=r_2=r_3$, the formula for the contracting similarity fixed points of the class of Self-affine fractal will become the formula for the contracting similarity fixed points of the self-similar fractal-----the general Sierpinski gasket[25].

The paper puts forward the formula for the contracting similarity fixed point of self-affine fractal, and the similar conclusions has not been found in the study of it by other scholars. The main result of this paper is the generalization of the formula for the contracting similarity fixed point of self-affine fractal. If the compression ratio of the self-affine fractal is consistent in all directions, the main conclusion of this paper will become the formula for the contracting similarity fixed point of the self-similar fractal. In other words, the formula for the contracting similarity fixed point of the self-similar fractal is just a special case of the conclusion in this paper.

By the Theorem 1.1, we have the following theorem.

Theorem

If $r_1=r_2=\cdots=r_k$, three vertices $(0,0)$, $(1/2\ ,\sqrt{3}/2)$, $(1,0)$ are the contracting similarity fixed points of any k-copy $E_{i_1 i_2\cdots i_k}$ of the class of self-affine fractal.

Proof

For any r_1, r_2, r_3,

$$0<r_1+r_2, r_2+r_3, r_3+r_1\le 1.$$

For any positive natural number $k\in N^+$, we have

(1)When $r_1=r_2=\cdots=r_k=1$,

$$(x,y)=(\frac{1}{2(1-r_{i_1}\cdots r_{i_n})}\sum_{k=1}^{n}(i_k-1)(1-r_{i_k})\frac{\sqrt{3}}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\ r_{i_1}\cdot r_{i_2}\cdots r_{i_{k-1}},$$

$$\frac{\sqrt{3}}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\sum_{k=1}^{n}D(i_k)(1-r_{i_k})r_{i_1}r_{i_2}\cdots r_{i_{k-1}})$$

$$=(\frac{1}{2(1-r_1^k)}\sum_{k=1}^{n}(1-1)(1-r_1)r_1\cdot r_1\cdots r_1,\ \frac{\sqrt{3}}{2(1-r_1^k)}\sum_{k=1}^{n}D(1)(1-r_1)r_1r_1\cdots r_1)$$

$$=(\frac{1}{2(1-r_1^k)}[(1-1)(1-r_1)+(1-1)(1-r_1)r_1+\cdots+(1-1)(1-r_1)r_1^{k-1}],$$

$$\frac{\sqrt{3}}{2(1-r_1^k)}[D(1)(1-r_1)+D(1)(1-r_1)r_1+\cdots+D(1)(1-r_1)r_1^{k-1}])$$

$$=(\frac{1}{2(1-r_1^k)}(0+\cdots+0),\frac{\sqrt{3}}{2(1-r_1^k)}(0+\cdots+0))$$

$$=(0,0).$$

(2)When $r_1=r_2=\cdots=r_k=2$

$$(x,y)=(\frac{1}{2(1-r_{i_1}\cdots r_{i_n})}\sum_{k=1}^{n}(i_k-1)(1-r_{i_k})\ r_{i_1}\cdot r_{i_2}\cdots r_{i_{k-1}},\ \frac{\sqrt{3}}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\sum_{k=1}^{n}D(i_k)(1-r_{i_k})r_{i_1}r_{i_2}\cdots r_{i_{k-1}})$$

$$=(\frac{1}{2(1-r_2^k)}\sum_{k=1}^{n}(2-1)(1-r_2)r_2\cdot r_2\cdots r_2,\ \frac{\sqrt{3}}{2(1-r_2^k)}\sum_{k=1}^{n}D(2)(1-r_2)r_2r_2\cdots r_2)$$

$$=(\frac{1}{2(1-r_2^k)}[(2-1)(1-r_2)+(2-1)(1-r_2)r_2+\cdots+(2-1)(1-r_2)r_2^{k-1}],$$

$$\frac{\sqrt{3}}{2(1-r_2^k)}[D(2)(1-r_2)+D(2)(1-r_2)r_2+\cdots+D(2)(1-r_2)r_2^{k-1}])$$

$$=(\frac{1}{2(1-r_2^k)}[(1-r_2)+(r_2-r_2^2)+\cdots+(r_2^{k-1}-r_2^k)],\ \frac{\sqrt{3}}{2(1-r_2^k)}[(1-r_2)+(r_2-r_2^2)+\cdots+(r_2^{k-1}-r_2^k)])$$

$$=(\frac{1}{2(1-r_2^k)}(1-r_2^k),\ \frac{\sqrt{3}}{2(1-r_2^k)}(1-r_2^k))$$

$$=(\frac{1}{2},\frac{\sqrt{3}}{2}).$$

(3)When $r_1=r_2=\cdots=r_k=3$

$$(x,y)=(\frac{1}{2(1-r_{i_1}\cdots r_{i_n})}\sum_{k=1}^{n}(i_k-1)(1-r_{i_k})\ r_{i_1}\cdot r_{i_2}\cdots r_{i_{k-1}},\ \frac{\sqrt{3}}{2(1-r_{i_1}r_{i_2}\cdots r_{i_n})}\sum_{k=1}^{n}D(i_k)(1-r_{i_k})r_{i_1}r_{i_2}\cdots r_{i_{k-1}})$$

$$=(\frac{1}{2(1-r_3^k)}\sum_{k=1}^{n}(3-1)(1-r_3)r_3\cdot r_3\cdots r_3,\ \frac{\sqrt{3}}{2(1-r_3^{\ k})}\sum_{k=1}^{n}D(3)(1-r_3)r_3r_3\cdots r_3)$$

$$=(\frac{1}{2(1-r_3^k)}[(3-1)(1-r_3)+(3-1)(1-r_3)r_3+\cdots+(3-1)(1-r_3)r_3^{k-1}],$$

$$\frac{\sqrt{3}}{2(1-r_3^k)}[D(3)(1-r_3)+D(3)(1-r_3)r_3+\cdots+D(3)(1-r_3)r_3^{k-1}])$$

$$=(\frac{1}{1-r_2^k}[(1-r_3)+(r_3-r_3^2)+\cdots+(r_3^{k-1}-r_3^k)],\ \frac{\sqrt{3}}{2(1-r_2^k)}(0+0+\cdots+0))$$

$$=(\frac{1}{1-r_2^k}(1-r_2^k),0)$$

$$=(1,0).$$

Since $k\in N^+$ is arbitrary, the theorem is proved.

EXAMPLE

As an important application, the article gives two practical examples. One of the examples is that self-affine fractal will become a special case---self-similar fractal when contracting ratios are consistent in all directions of self-affine fractal. Through two practical examples, the distribution of contracting similarity fixed points of self-affine fractal and self-similar fractal are presented in the paper. In addition, the paper gives specific graphics of the example. In view of the self-affine fractal self-similar fractal, the paper respectively marks the distribution of contracting similarity fixed points of first-level copy and second-level copy. According to the graphics, the distribution similarities and differences of the contracting similarity fixed points corresponding to the two kinds of fractals could be directly observed. In addition the basic conclusion is provided for the future research on contracting similarity fixed points of self-affine fractal and self-similar fractal.

Example

Let $r_1=1/2$, $r_2=1/4$, $r_3=1/3$,

If $n=1$, the contracting similarity fixed points of the class of self-affine fractal are $(0,0)$; $(1/2,\sqrt{3}/2)$; $(1,0)$.

If $n=2$, the contracting similarity fixed points of the self-affine fractal are $(0,0)$; $(3/14,3\sqrt{3}/14)$; $(2/5,0)$; $(3/7,3\sqrt{3}/7)$; $(1/2,\sqrt{3}/2)$; $(13/22,9\sqrt{3}/22)$; $(4/5,0)$; $(19/22,3\sqrt{3}/22)$; $(1,0)$.

If $n=3$, the contracting similarity fixed points of the self-affine fractal are $(0,0)$; $(1/10,\sqrt{3}/10)$; $(2/11,0)$; $(1/5,\sqrt{3}/5)$; $(15/62,15\sqrt{3}/62)$; $(13/46,\sqrt{3}/23)$; $(4/11,0)$; $(19/46,3\sqrt{3}/46)$; $(8/17,0)$; $(2/5,2\sqrt{3}/5)$; $(21/62,21\sqrt{3}/62)$; $(11/23,9\sqrt{3}/23)$; $(15/31,15\sqrt{3}/31)$; $(1/2,\sqrt{3}/2)$; $(16/17,0)$; $(1/2,45\sqrt{3}/94)$; $(13/23,9\sqrt{3}/23)$; $(8/11,0)$; $(14/17,0)$; $(55/94,39\sqrt{3}/94)$; $(43/70,27\sqrt{3}/70)$; $(35/46,3\sqrt{3}/46)$; $(19/23,3\sqrt{3}/23)$; $(1,0)$; $(79/94,15\sqrt{3}/94)$; $(61/70,9\sqrt{3}/70)$; $(67/70,3\sqrt{3}/70)$.

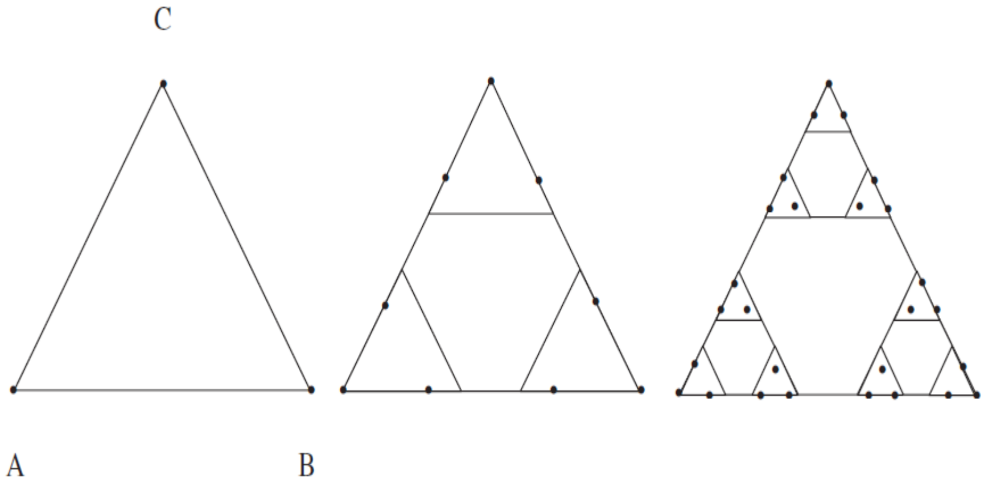

Figure 2 : The contracting similarity fixed points of the self-affine fractal

Example

Let $r = 0.25$, then the class of self-affine fractal become a special Sierpinski gasket.

If $n=1$, the contracting similarity fixed points of the special self-similar fractal are $(0,0)$; $(1/2, \sqrt{3}/2)$; $(1,0)$.

If $n=2$, the contracting similarity fixed points of the self-similar fractal are $(0,0)$; $(1/10, \sqrt{3}/10)$; $(1/5, 0)$; $(2/5, 2\sqrt{3}/5)$; $(1/2, \sqrt{3}/2)$; $(3/5, 2\sqrt{3}/5)$; $(4/5, 0)$; $(9/10, \sqrt{3}/10)$; $(1,0)$.

If $n=3$, the contracting similarity fixed points of the self-similar fractal are $(0,0)$; $(1/42, \sqrt{3}/42)$; $(1/21, 0)$; $(1/21, 2\sqrt{3}/21)$; $(5/42, 5\sqrt{3}/42)$; $(1/7, 2\sqrt{3}/21)$; $(4/21, 0)$; $(3/14, \sqrt{3}/42)$; $(5/21, 0)$; $(8/21, 8\sqrt{3}/21)$; $(17/42, 17\sqrt{3}/42)$; $(3/7, 8\sqrt{3}/21)$; $(10/21, 10\sqrt{3}/21)$; $(1/2, \sqrt{3}/2)$; $(11/21, 10\sqrt{3}/21)$; $(4/7, 8\sqrt{3}/21)$; $(1,0)$; $(25/42, 17\sqrt{3}/42)$; $(13/21, 8\sqrt{3}/21)$; $(18/21, 0)$; $(33/42, \sqrt{3}/42)$; $(17/21, 0)$; $(6/7, 2\sqrt{3}/21)$; $(37/42, 5\sqrt{3}/42)$; $(19/21, 2\sqrt{3}/21)$; $(20/21, 0)$; $(41/42, \sqrt{3}/42)$.

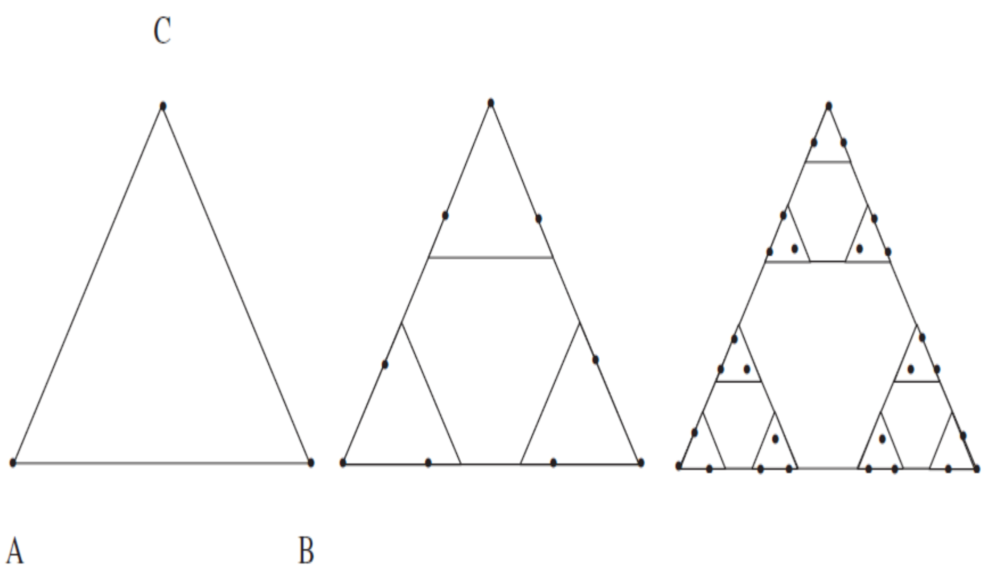

Figure 2 : The contracting similarity fixed points of the self-affine fractal

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