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Application of monte carlo method in numerical integral equations

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ABSTRACT

In this paper,we review the history of Monte Carlo methods. Then we study a numerical method bas-ed on Monte Carlo methods for the solution of integral equations.We give some numerical results. The results are supported by an application with Monte Carlo methods.Some concluding remarks are given According to the numerical results.

KEYWORDS

Monte caro methods;
Markov chain;
Integral equation;
Numerical experiments.

INTRODUCTION

During World war II,several famous physicists and mathematicians studied Monte Carlo methods such as J.von Neumann S.ulam, and E.Fermi. They worked for the United States Manhattan project[1]. In1948,N.Metropolis on the ENIAC first carried out actual Monte Carlo methods. In 1949 N.Metropolis and S.ulam published first paper on Monte Carlo methods. Monte Carlo methods had greatest influence on the science and engineering. Monte Carlo (MC) methods are stochastic techniques meaning. They are based on the use of random numbers and probability statistics to investigate problems.You can find MC methods used in everything from economics to nuclear physics to regulating the flow of traffic. Of course the way they are applied varies widely from field to field, and there are dozens of subsets of MC even within chemistry. But, strictly speaking, to call something a “Monte Carlo” experiment, all you need to do is use random numbers to examine some problem.

In 1950 Markovian chain,S.forsythe and R.Leibler[5] find the inverse of a matrix by using Monte Ca-rlo methods. Since then,many numerical algorithms based on Monte Carlo methods have been are ap-plied varies widely from field to field.

In this paper,we mainly study the Monte Carlo methods in solving integral equations. Our idea for solving integral by Monte Carlo method is using Markov chain with State space. In section 2 we review briefly the Monte Carlo methods. In section 3 and section 4 we use Monte Carlo methods to evaluate integrals. We give numerical experiments and a summary remarks in section 6.

MONTE CARLO METHODS

Assume that $X_{1,...,}X_n$ are i.i.d. $f(x|\theta), x \in \chi$, with $E(g(X_i)) = \lambda_i$, for all $i = 1.2...,n$ .Then the Strong Law of Large Numbers $(S.L.L.N)$ tells as with probability 1 we have

$$\frac{1}{n}\sum_{i=1}^{n} g(x_i) \to \lambda, \text{ as } n \to \infty .$$

The SLLN states that if we where to generate a large number of values from $f(x|\theta)$,then in order to approximate $\lambda = E(g(X))$,all wo have to do is take average of the generated values evaluated through $g(x)$.

The value $g(x_1),...,g(x_n)$ can be thought of as realizations from the random variable .

The Monte Carlo approach consists of using this idea to evaluate the integral $E(g(X)) = \lambda$

$$\lambda = E(g(X)) = \int_{\chi} g(x)f(x)dx \approx \frac{1}{n}\sum_{i=1}^{n} g(X_i),$$

where $X_{1,...,}X_n$ a random sample from $f(x|\theta)$.

USING MONTE CARLO METHODS TO EVALUATE INTEGRALS

Suppose we want to evaluate the integral $\int_{\chi} h(x)dx$, where $h(x)$ doesn't have to be a p.d.f.,and χ could be the whole real line. Depending on the form of χ, we have different ways of attacking the pro-blem.First wo present transformation to the interval and then a general method without transforming.

Evaluating an integral over $\chi = [0,1]$

To evaluate $\int_0^1 h(x)dx$ we write

$$\int_0^1 h(x)dx = \int_0^1 h(x)1(0 \le x \le 1)dx,$$

And hence the algorithm becomes:

Step 1: Generate $U_1,...,U_L i,i,d.U(0,1)$ for a very large L.

Step 2: Calculate $\int_0^1 h(x)dx \approx \frac{1}{L}\sum_{i=1}^{L} h(U_i).$

Evaluating an integral over $\chi = [a,b]$

To evaluate $\int_a^b h(x)dx$ we take the transformation

$$y = \frac{x-a}{b-a} \Rightarrow x = (b-a)y + a,\ dx = (b-a)dy$$

to obtain $\int_a^b h(x)dx = \int_0^1 (b-a)h((b-a)y+a)dy,$ and hence we can use the algorithm in section 3.

Evaluating an integral over $\chi = [a,+\infty]$

To evaluate $\int_a^{+\infty} h(x)dx$, we take the transformation

$$y = \frac{1}{x-a+1} \Rightarrow x = \frac{1}{y} + a - 1, dx = -\frac{1}{y^2}dy,$$

To obtain

$$\int_a^{+\infty} h(x)dx = \int_0^1 \frac{1}{y^2} h(\frac{1}{y} + a - 1)dy,$$

and hence we can use the algorithm in section 3.1.

Evaluating an integral over $\chi = [-\infty,b]$.

To evaluate $\int_{-\infty}^{b} h(x)dx$, we take transformation

$$y = \frac{1}{b-x+1} \Rightarrow x = -\frac{1}{y} + b + 1, dx = \frac{1}{y^2}dy,$$

To obtain

$$\int_{-\infty}^{b} h(x)dx = \int_0^1 \frac{1}{y^2} h(-\frac{1}{y} + b + 1)dy,$$

and hence we can use the algorithm in section 3.1.

Evaluating an integral over $\chi = [-\infty,+\infty]$

To evaluate $\int_{-\infty}^{+\infty} h(x)dx$ we write it as

$$\int_{-\infty}^{+\infty} h(x)dx = \int_{-\infty}^{0} h(x)dx + \int_{0}^{+\infty} h(x)dx,$$

and we fall in the previous cases.For the first integral $\int_{-\infty}^{0} h(x)dx$, we take the transformation

$$y = \frac{1}{1-x} \Rightarrow x = -\frac{1}{y} + 1, dx = \frac{1}{y^2}dy,$$

and for the second $\int_0^{+\infty} h(x)dx,$ the transformation

$$y = \frac{1}{x+1} \Rightarrow x = \frac{1}{y} - 1, dx = -\frac{1}{y^2}dy,$$

and thus obtain $\int_{-\infty}^{+\infty} h(x)dx = \int_0^1 \frac{1}{y^2} h(-\frac{1}{y}+1)dy + \int_0^1 \frac{1}{y^2} h(\frac{1}{y}-1)dy = \int_0^1 \frac{1}{y^2}[h(-\frac{1}{y}+1) + h(\frac{1}{y}-1)]dy$

and hence we can use the algorithm in section 3.1on those integrals.

MULTIVARIATE EXTENSIONS:USING TRANSFORMATIONS TO [0,1]

To evaluate the integral $\int_{\chi} h(x_1, x_2, ..., x_p)dx_1dx_2...dx_p$, where the set is separable,i.e.,we simply write

$$\int_{\chi} h(x_1, x_2, ..., x_p)dx_1dx_2...dx_p = \int_{x_1}\int_{x_2}...\int_{x_p} h(x_1,x_2,...,x_p)dx_1...dx_p$$

and then take appropriate transformation depending on form of the $\chi_i{}'s$. For instant if $\chi = \chi_1 \times \chi_2 = R^2$, then we would write

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x_1,x_2)dx_1dx_2 = \int_{-\infty}^{+\infty}[\int_{-\infty}^{0} h(x_1,x_2)dx_1 + \int_{0}^{+\infty} h(x_1,x_2)dx_1]dx_2$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} h(x_1,x_2)dx_1dx_2 + \int_{-\infty}^{0}\int_{0}^{+\infty} h(x_1,x_2)dx_1dx_2$$

$$+ \int_{0}^{+\infty}\int_{-\infty}^{0} h(x_1,x_2)dx_1dx_2 + \int_{0}^{+\infty}\int_{0}^{+\infty} h(x_1,x_2)dx_1dx_2$$

$$= \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2} h(-\frac{1}{y}+1, -\frac{1}{y}+1)dy_1dy_2 + \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2} h(-\frac{1}{y}+1, \frac{1}{y}-1)dy_1dy_2$$

$$+ \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2} h(\frac{1}{y}-1, -\frac{1}{y}+1)dy_1dy_2 + \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2} h(\frac{1}{y}-1, \frac{1}{y}-1)dy_1dy_2$$

$$= \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2}[h(-\frac{1}{y}+1, -\frac{1}{y}+1) + h(-\frac{1}{y}+1, \frac{1}{y}-1)]dy_1dy_2$$

$$+ \int_0^1\int_0^1 \frac{1}{y_1^2 y_2^2}[h(\frac{1}{y}-1, -\frac{1}{y}+1) + h(\frac{1}{y}-1, \frac{1}{y}-1)]dy_1dy_2$$

Multivariate extensions:No transformations

Consider now the integral

$$\int_{\chi} h(x_1, x_2, ..., x_p)dx_1dx_2...dx_p = \int_{\chi_1}\int_{\chi_2}...\int_{\chi_p} h(x_1, x_2, ..., x_p)dx_1dx_2...dx_p$$

$$= \int_{\chi_1}\int_{\chi_2}...\int_{\chi_p} \frac{h(x_1,x_2,...,x_p)}{f(x_1,x_2,...,x_p)} f(x_1,x_2,...,x_p)dx_1dx_2...dx_p$$

Where

$$g(x_1,x_2,...,x_p) = \frac{h(x_1,x_2,...,x_p)}{f(x_1,x_2,...,x_p)},$$

and $f(x_1,x_2,...,x_p)$ is a multivariate p.d.f.with support $\chi = \chi_1 \times \chi_2 \times ... \times \chi_p$, that is known and can be used to given us a random sample of random vectors,namely

$$\chi_1 = (x_{11}, x_{12}, ..., x_{1p}),$$
$$\chi_2 = (x_{21}, x_{22}, ..., x_{2p}),$$
$$...$$
$$\chi_n = (x_{n1}, x_{n2}, ..., x_{np}).$$

Then the integral is approximated by

$$\int_{\chi} h(x_1,x_2,...,x_p)dx_1dx_2...dx_p$$
$$\approx \frac{1}{n}\sum_{i=1}^{n} g(x_i)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \frac{h(x_1,x_2,...,x_p)}{f(x_1,x_2,...,x_p)}$$

Now for the case where χ imposes relationships on the vector χ. Here

$$\chi = \{x = (x_1,...,x_p) \in R^p : x_i = q_i(x_1,x_2,...,x_p), i = 1,2,...,p\}.$$

We can easily bring this into one of the previous forms by considering

$$\int_{\chi} h(x_1,x_2,...,x_p)dx_1dx_2...dx_p$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} 1(x \in \chi)h(x_1,x_2,...,x_p)dx_1dx_2...dx_p$$

where 1()the indicator of the set

Numerical experiments

Example 1

Consider

$$I_y = \int_A x^2 dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} hx^2dx = \frac{b^3h}{12}$$

for which the exact solution is 0.1667 when b=1,h=2.

TABLE 1 : Inertial moment results and relative errors of Monte Carlo

N	Results	relative errors
10	0.1500	10.00
100	0.1630	2.220
1000	0.1645	1.320
10000	0.1651	0.960

We can easily see that

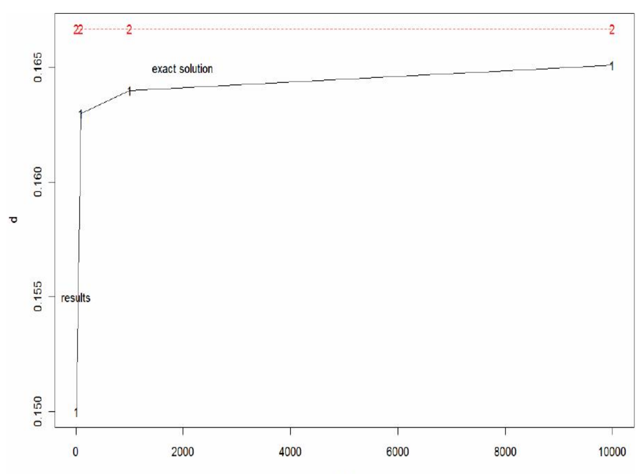

Figure 1 : Inertial moment results and relative errors of Monte Carlo

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$$D(I_y) \approx \widehat{D}(I_y) = \frac{I_y(1-I_y)}{N}$$

$$\sigma(I_y) \approx \widehat{\sigma}(I_y) = \sqrt{\frac{I_y(1-I_y)}{N}}$$

TABLE 2 : Inertial moment results and variance of Monte Carlo

N	Results	variance
10	0.1500	0.00225
100	0.1630	0.00026569
1000	0.1645	0.0000270
10000	0.1651	0.000002725801

TABLE 3 : Inertial moment results and standard variance of Monte Carlo

N	Results	Standard variance
10	0.1500	0.047434
100	0.1630	0.0163
1000	0.1645	0.005196
10000	0.1651	0.001649

Example 2

Consider $\int_0^1 e^{x-1}dx$ the exact solution is 0.6.

TABLE 4 : Inertial moment results and variance of Monte Carlo

Random r_1		Random r_2		$g(r_1)=e^{r_1-1}$		Number ($r_2 \le g(r_1)$)	
0.266	0.914	0.742	0.791	0.480	0.918	0	1
0.606	0.925	0.751	0.295	0.674	0.928	0	1
0.454	0.417	0.936	0.121	0.579	0.558	0	1
0.698	0.252	0.480	0.781	0.730	0.473	1	0
0.037	0.916	0.863	0.115	0.382	0.919	0	1
0.846	0.001	0.576	0.847	0.857	0.368	1	0
0.915	0.532	0.496	0.732	0.919	0.626	1	0
0.107	0.255	0.493	0.174	0.409	0.475	0	1
0.548	0.011	0.196	0.121	0.636	0.372	1	1
0.489	0.835	0.495	0.155	0.600	0.848	1	1
sum				12.76		12	
mean				0.638		0.60	

CONCLUSION

The present study successfully applied a numerical algorithm with Monte Carlo method to solve integral equation. It can be seen that the proposed method is efficient and accurate to estimate the solution. Furthermore the results indicate that the present Monte Carlo method is preferable when one needs to have a rough estimation.

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