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APPLICATION OF DIFFERENTIAL TRANSFORM METHOD FOR SOLVING DIFFERENTIAL AND INTEGRAL EQUATIONS

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ABSTRACT

Differential equation describes exchange of matter, energy, information or any other quantities; often as they vary in time and /or space. Integral transform which is particularly useful in solving differential equations. Integral transform is a very powerful mathematical tool applied in various areas of engineering and science. Using differential transformation method (DTM) to solve differential equation of the type Lane Emden equations as singular initial value problems. Also DTM was employed to solve Volterra integral equations of second kind. This paper will discuss the application of the differential transformation method, which is based on Taylor series expansion, to construct analytical approximate solution of the initial value problems.

Key words: Differential transformation method, Lane – Emden differential equation, systems of Volterra integral equations of the second kind.

INTRODUCTION

Integral transform is a very powerful mathematical tool applied in various areas of engineering and science. Historically, the concept of an integral transform originated from the celebrated Fourier integral formula. Laplace transform –

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt \quad , s = \sigma + j\omega \qquad \ldots(1)$$

named after the mathematician Pierre-Simon Laplace (1749–1827), is a major mathematical tool for the transformation of a continuous-time function to the complex s plane. Fourier transform –

$$F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt \qquad \ldots(2)$$

named after Jean Baptiste Joseph Fourier (1768–1830), a scientist adviser in Napoleon Bonaparte's army,

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may be viewed as a special case of the Laplace transform, where the complex frequency variable $s = \sigma + j\omega$ is limited to the imaginary line $s = j\omega$[1]. The connection between the Fourier transform and the Laplace transform is intimate, but they are not equivalent. For example, the step function has a Laplace transform, but not a Fourier transform. And while the Fourier transform is useful in finding the steady-state output of a linear circuit in response to a periodic input, the Laplace transform can provide both the steady-state and transient responses for periodic *and* aperiodic inputs[2].

Laplace transform has been considered as a useful tool to solve integer-order or relatively simple fractional-order differential equations. Inverse Laplace transform is an important but difficult step in the application of Laplace transform. Differential equations describe exchanges of matter, energy, information or any other quantities, often as they vary in time and/or space. Their thorough analytical treatment forms the basis of fundamental theories in mathematics. [For a complicated differential equation, however, it is difficult to analytically calculate the inverse Laplace transformation. So, the numerical inverse Laplace transform algorithms are often used to calculate the numerical results.In mathematics, an integral equation is an equation in which an unknown function appears under an integral sign. There is close connection between differential and integral equations[3,4].

Historical notes

The concept of the differential transform method was first proposed by Zhou (1986) and has been used extensively during the last two decades to solve effectively and easily various both linear and nonlinear initial value problems, in electric circuit analysis[5-7]. The main advantage of these methods is that they can be applied directly to differential equations without requiring linearization, discretization or perturbation.

Basic Ideas: Differential transform method (DTM)

The basic definitions and fundamental operations of the differential transform are defined as follows (Chen and Ho, 1996, 1999). The differential transform of the function $u(x)$ is defined as,

$$T\{x(k)\} = X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \qquad \ldots(3)$$

Where $x(t)$ is the original function and $X(k)$ is the transformed function. Differential inverse transform of $X(k)$, is defined as –

$$T^{-1}\{X(k)\} = x(t) = \sum_{k=0}^{\infty} X(k)(t-t_0)^k \qquad \ldots(4)$$

When k = 0, 0 $t = t$, the function $x(t)$ defined as (5), is expressed as the following –

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right] t^k \qquad \ldots(5)$$

In real applications when the general term of the series cannot be recognized, a truncated series can be considered. Equation (6) implies that, the concept of one dimensional differential transform is almost the same as one-dimensional Taylor series expansion. In this study use lower case letters to present the original functions and upper case letters stand for the transformed functions (T-functions). From definition (4 to 6), one can easily prove that the transformed functions comply with the following basic mathematical operations[5,6].

Some of the fundamental mathematical operations performed by differential transform method are listed in Table 1.

Table 1: The fundamental operations of one-dimensional DTM

S. No.	Original function x (t)	Transformed function X (k)
1	$\alpha\, x(t) \pm \beta\, y(t)$	$\alpha\, X(k) \pm \beta\, Y(k)$
2	$\frac{d^m x(t)}{dt^m}$	$\frac{(k+m)!}{k!} X(k+m)$
3	$x(t)y(t)$	$\sum_{l=0}^{k} X(l)Y(k-l)$
4	$x(t)y(t)z(t)$	$\sum_{l=0}^{k} \sum_{m=0}^{k-l} X(l)Y(k-m)Z(k-l-m)$
5	$\int_0^t y(t)dt$	$\frac{Y(k-1)}{k}$ where $k \geq 1$ and $Y(0) = 0$
6	$\int_0^t y_1(t)y_2(t)\ldots\ldots y_n(t)dt$	$\frac{1}{k} \sum_{k_{n-1}=0}^{k-1} \sum_{k_{n-2}=0}^{k_{n-1}-1} \ldots\ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y_1(k_1)Y_2(k_2-k_1)\ldots\ldots Y_n(k-k_{n-1}-1)$ where $k \geq 1$ and $Y(0) = 0$
7	$x(t)\int_0^t y(t)dt$	$\sum_{l=1}^{k} X(k-1)\frac{Y(l-1)}{l}, k \geq 1$
8	$x_1(t)x_2(t)\ldots\ldots x_n(t)\int_0^t y_1(t)y_2(t)\ldots\ldots y_m(t)dt$	$\sum_{k_{m+n}=1}^{k-1} \sum_{k_{m+n-2}=1}^{k_{n+m-1}} \ldots\ldots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} Y_1(k_1)Y_2(k_2-k_1)\ldots\ldots Y_m(k_m-k_{m-1}).$ $X_1(k_{m+1}-k_m)X_2(k_{m+2}-k_{m+1})\ldots X_n(k-k_{n+m-1}), \quad k \geq 1$
9	t^m	$\delta(k-m) = 1$ if $k = m, 0$ if $k \neq m$
10	$\exp(t)$	$\frac{1}{k!}$
11	$\sin(\lambda t + \omega)$	$\frac{\lambda^k}{k!}\sin\left(\frac{k\pi}{2} + \omega\right)$
12	$\cos(\lambda t + \omega)$	$\frac{\lambda^k}{k!}\cos\left(\frac{k\pi}{2} + \omega\right)$

Lane –Emden equation

Singular initial value problems in the second order ordinary differential equations occur in several models of mathematical physics and astrophysics such as the theory of stellar structure the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermo ionic currents which are modeled by means of the following Lane–Emden equation [6].

$$x''(t) + \frac{\alpha}{t} x'(t) + f(t,x) = g(t), \quad 0 < t \leq 1, \alpha \geq 0 \qquad \ldots(6)$$

under the following initial conditions.

x'' (0) = A, x' (0) = B …(7)

where A and B are constants, f(t, x) is a continuous real valued function and g(t)€C[0,1].

Numerical example

Example 1: We first start by considering the following Lane-Emden equation given in[6] -

$$x''(t)+\frac{8}{t}x'(t)+t\,x(t)=t^5-t^4+44t^2-30t\,,\ 0\prec t\leq 1, \quad \ldots(8)$$

with initial conditions-

x (0) = 0, x' (0) = 0 …(9)

By multiplying both sides of Eq. (9) by t and then taking differential transformation of both sides of the resulting equation using above table 1, the following recurrence relation is obtained: Differential transformation method for solving differential equations –

$$X(k+1)=\frac{1}{(k+1)(k+8)}(\ \delta(k-6))-\delta(k-5)+44\delta(k-3)-30\delta(k-2)-\sum_{l=0}^{k}\delta(l-2)X(k-1) \quad \ldots(10)$$

By using Eqs. (4) and (10), the following transformed initial conditions at 0 0 *t* can be obtained:

X (0) = 0 …(11)

X (1) = 0 …(12)

Substituting Eqs. (10) and (11) at *k* = 1 into Eq. (11), we have

X (2) = 0 …(13)

Following the same recursive procedure, we find X (k + 1) – 0, k = 4, 5…..and listing the computation and result corresponding to *n* = 4 , we have

X (3) = -1, X (4) = 1 …(14)

Using Eqs. (12)-(15) and the inverse transformation rule in Eq. (6), we get the following solution:

X (t) = $t^4 - t^3$ …(15)

Note that for *n* > 4 one evaluates the same solution, which is the exact solution of Eq.(7) with the initial conditions in Eq. (8).

A system of volterra integral equations of the second kind

Differential transform method leads to an iterative procedure for obtaining an analytic series solutions of functional equations. In recent years, applications of differential transform theory have appeared in many researches (Arikhoglu and Ozkol, 2007, 2006, 2005; Ayaz, 2003; Chen and Ho, 1996, 1999; Hassan and Abdel-Halim, 2008). A system of Volterra integral equations of the second kind can be presented as the following[7]:

$$F(t)=G(t)+\int_0^t K(t,s,F(s))ds, \quad \ldots(16)$$

Where F(*t*) = $[f_1(t), f_2(t)\ldots\ldots, f_n(t)]^T$

G(t) = $[g_1(t), g_2(t)\ldots\ldots, g_n(t)]^T$

$K(t, s, F(s)) = [k_1(t, s, F(s)), k_2(t, s, F(s)), \ldots\ldots., k_n(t, s, F(s))]^T$

Numerical example

Example 2: Consider the following non- linear system of volterra integral equations of the second kind –

$$\begin{cases} f_1(t) = \sin t - t - \int_0^t \left(f_1^2(s) + f_2^2(s)\right)ds \\ f_2(t) = \cos t - \frac{1}{2}\sin^2 t + \int_0^t f_1(s)f_2(s)ds \end{cases} \quad \ldots(17)$$

One can readily find the differential transform of (17) as follows:

$$F_1(k) = \frac{1}{k!}\sin\frac{k\pi}{2} - \delta(k-1) + \frac{1}{k}\sum_{k_1=0}^{k-1} F_1(k_1)F_1(k-k_1-1) + \frac{1}{k}\sum_{k_1=0}^{k-1} F_2(k_1)F_2(k-k_1-1)$$

$$F_2(k) = \frac{1}{k!}\cos\frac{k\pi}{2} - \frac{1}{2}\sum_{K_1=0}^{k}\frac{1}{k!}\sin\left(\frac{k_1\pi}{2}\right)\frac{1}{(k-k_1)!}\sin\left(\frac{\pi(k-k_1)}{2}\right) + \frac{1}{k}\sum_{k_1=0}^{k-1} F_1(k_1)F_2(k-k_1-1) \quad \ldots(18)$$

Consequently,

$$F_1(0) = 0, F_1(1) = -1, F_1(2) = 0, F_1(3) = -\frac{1}{6} \quad \ldots(19)$$

$$F_2(0) = 1, F_2(1) = -\frac{1}{2}, F_2(2) = 0, F_2(3) = \frac{1}{4!} \quad \ldots(20)$$

Closed forms of the solution in this example are as the following**:**

$$f_1(t) = \sum_{k=0}^{\infty} F(k)t^k. = \sin t, f_2(t) = \sum_{k=0}^{\infty} G(k)t^k. = \cos t \quad \ldots(21)$$

In this example, we have derived an exact solution.

CONCLUSION

In this paper, we extend the application of the differential transformation method, which is based on Taylor series expansion, to construct analytical approximate solutions of the initial value problem. In this paper, differential transform method is implemented to the Lane-Emden differential equations singular initial value problems. Also differential transform method has been used successfully for finding the solutions of non-linear systems of Volterra integral equations of the second kind. Differential transform method is a very powerful and efficient technique, for finding exact solutions for a wide class of problems.

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