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A new algorithm with mixed finite element method for semilinear parabolic equations

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ABSTRACT

A kind of two-dimensional semilinear parabolic equation is presented by two-grid methods for nonconforming mixed finite element. The corresponding convergence analysis is presented and the error estimates are obtained by use of the interpolation operator instead of the conventional elliptic projection which is an indispensable tool in the convergence analysis. Compared with the previous literature which was solved by conforming mixed finite element, the same order of convergence is obtained and the method can be parallelized in a highly efficient manner.

KEYWORDS

Nonconforming;
Mixed finite element;
Semilinear parabolic equation;
Two-grid method;
Error estimates.

INTRODUCTION

Parabolic equation is considered a prototype of a model that arises in flow through porous media. There have been many numerical modeling methods for parabolic equations. For example[1-4]: used the mixed finite element method for parabolic equations[5] studied the parabolic equation with anisotropic mixed finite element method. In[6], the maximum norm estimates of mixed finite element method for parabolic equations is obtained. Expanded mixed finite element method were used in[7] and H1-Galerkin mixed finite element methods were studied in[8,9].

In this paper, we will study a class of semilinear parabolic equations with two-grid mixed finite element method. The two-grid method is a method by which the nonlinear system is only executed on the coarse grid of size H and then the linear system is solved on the fine grid of size h. Inspired by Xu[10,11] for a method to solve nonsymmetric and indefinite linear algebraic systems, we employ two finite element spaces: V_H and V_h in our discrete schemes. On the coarser space V_H, we use the standard finite element discretization to obtain a rough approximation $u_H \in V_H$ and then solve a linearized equation based on u_H to produce a corrected solution $u_h \in V_h$. A remarkable fact about this simple technique is that the space V_H can be extremely coarse (in contrast to V_h) and still maintain the optimal accuracy. The method can not only accelerate the convergence but also improve the computational efficiency. There are a lot of other problems which were solved by the two-grid method. For example:[12] studied the reaction-diffusion equations and[13] studied the stream function form of Navier-Stokes equations with the method[14] studied the convection-dominated diffusion equation with two-grid characteristics finite element method[15] studied a kind of parabolic problems and[16] studied another parabolic problems with the two-grid method. However[15], and[16] were studied for the conforming finite element. As we all know, the nonconforming elements whose degree defined on the edges of the element and in the element itself, it can lead to cheap local communication and the method can be parallelized in a highly efficient manner. Compared with the general elements, this method can improve the efficiency of calculation.

In this paper, we will apply a kind of nonconforming element to the parabolic equations with two-grid method. In the first step, we got the error estimate $O(\Delta t^2 + H^3)$ on the coarse grid of size H which is the same order of convergence as[15], and in the second step, we got the error estimate $O(\Delta t^2 + h^3 + H^5)$ on the fine grid of size h. The error analysis demonstrates that if the mesh size of coarse grid equal to $\sqrt[5]{h^3}$, the two-grid solution in the second step and the finite element solution in the first step have the same order of accuracy. We also used interpolation instead of Fortin projection in[15] and improved the computational efficiency.

CONSTRUCTION OF THE ELEMENT AND DISCRETE SCHEMES

Now let us consider the following semilinear parabolic equations:

$$\begin{cases} p_t - \Delta p = f(p), in \quad \Omega\times(0,T) \\ p = 0, \quad on \quad \partial\Omega\times(0,T) \\ p(X,0) = p_0(X) \ , \ in \quad \Omega \end{cases} \tag{1}$$

For the sake of convenience, Let $\Omega \in R^2$ be a convex polygon domain composed by a family of rectangular meshes T_h, which does not need to satisfy the regular conditions. ∂ is the boundary of the domain Ω and Δ is the Laplace operator. $f(p)$ satisfies the Lipschitz condition. For all $K \in T_h$, denoted the center of element K by (x_K, y_K), and the length of edges parallel to x-axis and y-axis by $2h_x$, $2h_y$, respectively. $\mathbf{Z}_1(\mathbf{x}_K - \mathbf{h}_x, \mathbf{y}_K - \mathbf{h}_y)$, $\mathbf{Z}_2(\mathbf{x}_K + \mathbf{h}_x, \mathbf{y}_K - \mathbf{h}_y)$, $\mathbf{Z}_3(\mathbf{x}_K + \mathbf{h}_x, \mathbf{y}_K + \mathbf{h}_y)$ and $\mathbf{Z}_4(\mathbf{x}_K - \mathbf{h}_x, \mathbf{y}_K + \mathbf{h}_y)$ are the four vertices, and $\mathbf{l}_i = \mathbf{Z}_i\mathbf{Z}_{i+1}(\mathbf{mod(4)})$ are the four edges. Let $\hat{K}$ be the reference element, the four vertices are $\hat{\mathbf{a}}_1(-1,-1)$, $\hat{\mathbf{a}}_2(1,-1)$, $\hat{\mathbf{a}}_3(1,1)$ and $\hat{\mathbf{a}}_4(-1,1)$. Let $\hat{\mathbf{l}}_i = \hat{\mathbf{a}}_i\hat{\mathbf{a}}_{i+1}$ $(\mathbf{mod(4)})$, then there exists an reversible mapping $\mathbf{F}_K : \hat{\mathbf{K}} \to \mathbf{K}$:

$$\begin{cases} \mathbf{x} = \mathbf{x}_K + \mathbf{h}\boldsymbol{\xi}, \\ \mathbf{y} = \mathbf{y}_K + \mathbf{h}\boldsymbol{\eta}. \end{cases}$$

We define the new finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ as follows:

$\Sigma^1 = \{\hat{v}_i, i = 1,2,3,4\}$, $\hat{P}_1 = span\{1, \xi, \eta, \eta^2\}$ and

$\Sigma^2 = \{\hat{v}_i, i = 1,2,3,4\}$, $\hat{P}_2 = span\{1, \xi, \eta, \xi^2\}$, where

$\hat{v}_i = \frac{1}{\hat{l}_i}\int_{\hat{l}_i} \hat{v} d\hat{s}, i = 1,2,3,4.$

It can be checked that the above interpolation is well-posed, the interpolation function can be expressed as follow

$$\hat{I}^1\hat{v}^1 = \frac{3}{4}(\hat{v}_2 + \hat{v}_4) - \frac{1}{4}(\hat{v}_1 + \hat{v}_3) + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta + \frac{3}{4}(\hat{v}_1 - \hat{v}_2 + \hat{v}_3 - \hat{v}_4)\eta^2.$$

$$\hat{I}^2\hat{v}^2 = \frac{3}{4}(\hat{v}_1 + \hat{v}_3) - \frac{1}{4}(\hat{v}_2 + \hat{v}_4) + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta + \frac{3}{4}(-\hat{v}_1 + \hat{v}_2 - \hat{v}_3 + \hat{v}_4)\xi^2.$$

The associated finite element space V_h and W_h is defined by $V_h = \{v_h; v_h^{\ j} \in \hat{P}, \int[v_h] = 0\}$, $W_h = \{q_h, q_h|_K$ *is a cons* tan t), where $[v]$ denote the jump value of v across the boundary F and $[v] = v$ when $F \in \partial\Omega$.

ERROR ESTIMATES

Introducing the auxiliary variable $q = \nabla p$, and rewriting the equation (1) as:

$$\begin{cases} (p_t, w) + (\nabla\cdot u, w) = (f(p), w), \forall w \in W \\ (u, v) - (p, \nabla\cdot v) = 0, \forall v \in V \end{cases} \tag{2}$$

The discrete problem of (2) reads as: find $\{p_h, u_h\} : [0,T] \to W_h \times V_h$, such that

$$\begin{cases} (p_{ht}, w_h) + (\nabla\cdot u_h, w_h) = (f(p_h), w_h), \forall w_h \in W_h \\ (u_h, v_h) - (p_h, \nabla\cdot v_h) = 0, \forall v_h \in V_h \end{cases} \tag{3}$$

By the theory of differential equations, (3) has a unique solution.

TWO-GRID ALGORITHM AND ERROR ESTIMATES

We will give the two-grid algorithm which solved a nonlinear system of equations on the coarse grid and then made linear modification on the fine grid. Let $\Delta t = \frac{T}{N}, t_n = n\Delta t$, The discrete problem of (2) reads as: Find $\{p_h^n, u_h^n\}:[0,T]\to W_h\times V_h$ such that:

$$\begin{cases}\frac{1}{\Delta t}(p_h^n - p_h^{n-1}, w_h) + (\nabla\cdot u_h^{n-\frac{1}{2}}, w_h) = (f(p_h^{n-\frac{1}{2}}), w_h), \ \forall\, w_h\in W_h,\\ (u_h^{n-\frac{1}{2}}, v_h) - (p_h^{n-\frac{1}{2}}, \nabla\cdot v_h) = 0, \quad \forall\, v_h\in V_h\end{cases}$$

First of all, we will introduce the mixed element space $W_H\times V_H(\subset W_h\times V_h)$ which was defined on the coarse quasiuniform rectangular subdivision and the mesh size is H $(h \ll H<1)$, so the two-grid algorithm is listed as follows:

Step 1: Solve a nonlinear system $\{p_H^n, u_H^n\}:[0,T]\to W_H\times V_H$ on the coarse grid

$$\begin{cases}\frac{1}{\Delta t}(p_H^n - p_H^{n-1}, w_H) + (\nabla\cdot u_H^{n-\frac{1}{2}}, w_H) = (f(p_H^{n-\frac{1}{2}}), w_H),\\ \forall\, w_H\in W_H,\\ (u_H^{n-\frac{1}{2}}, v_H) - (p_H^{n-\frac{1}{2}}, \nabla\cdot v_H) = 0, \quad \forall\, v_H\in V_H\end{cases}$$

Step 2:Solve a linear system $\{p_h^n, u_h^n\}:[0,T]\to W_h\times V_h$ on the fine grid

$$\begin{cases}\frac{1}{\Delta t}(p_h^n - p_h^{n-1}, w_h) + (\nabla\cdot u_h^{n-\frac{1}{2}}, w_h) = (f(p_H^{n-\frac{1}{2}})\\ + f_p(p_H^{n-\frac{1}{2}})(p_h^{n-\frac{1}{2}} - p_H^{n-\frac{1}{2}}), w_h), \ \forall\, w_h\in W_h,\\ (u_h^{n-\frac{1}{2}}, v_h) - (p_h^{n-\frac{1}{2}}, \nabla\cdot v_h) = 0, \quad \forall\, v_h\in V_h\end{cases}$$

For the convenience of error estimates, we suppose that $\|f_u\|_\infty = F, \|f_{uu}\|_\infty = G$. Let $\Delta t>0, N=\frac{T}{\Delta t}$, and $t^n = n\Delta t$.

Theorem 1 under the coarse grid subdivision, $(u_H^n, p_H^n)\in V_H\times W_H$ is the solution of step 1, there is constant and independent of the H, there exists a constant C which is independent of the mesh size, there hold

$$\|u^m - u_H{}^m\| + \left[\sum_{n=1}^m \Delta t\left\|p^{n-\frac{1}{2}} - p_H^{n-\frac{1}{2}}\right\|^2\right]^{\frac{1}{2}} \le c(\Delta t^2 + H^3)$$

where $1\le m\le N$.

Proof : Let

$$u - u_H = (u - I_H u) + (I_H u - u_H) = \xi + \eta,$$

$$p - p_H = (p - \Pi_H p) + (\Pi_H p - p_H) = \rho + \theta .$$

We can get the error equation from step 1

$$(u_t^{n-\frac{1}{2}} - \frac{1}{\Delta t}(u_H^n - u_H^{n-1}), \eta^{n-\frac{1}{2}}) - (p^{n-\frac{1}{2}} - p_H^{n-\frac{1}{2}}, \nabla\eta^{n-\frac{1}{2}})$$

$$= (f(u^{n-\frac{1}{2}}) - f(u_H^{n-\frac{1}{2}}), \eta^{n-\frac{1}{2}}) \quad \textbf{(4)}$$

We choose $v_H = \eta^{n-\frac{1}{2}}, w_H = \nabla\eta^{n-\frac{1}{2}}$.

Not that

$$u_t^{n-\frac{1}{2}} - \frac{1}{\Delta t}(u^n - u^{n-1}) = \frac{1}{2\Delta t}(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}}(\tau - t_{n-1})^2 u_{ttt}(\tau)d\tau$$

$$+\int_{t_{n-\frac{1}{2}}}^{t_n}(\tau - t_n)^2 u_{ttt}(\tau)d\tau),$$

Let $u_t^{n-\frac{1}{2}} = \frac{1}{\Delta t}(u^n - u^{n-1}) + \varepsilon$,

where

$$\varepsilon = \frac{1}{2\Delta t}(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}}(\tau - t_{n-1})^2 u_{ttt}(\tau)d\tau + \int_{t_{n-\frac{1}{2}}}^{t_n}(\tau - t_n)^2 u_{ttt}(\tau)d\tau)$$

The equation (4) can be written that:

$$\frac{1}{\Delta t}(\eta^n - \eta^{n-1}, \eta^{n-\frac{1}{2}}) + (\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}})$$

$$= -\frac{1}{\Delta t}(\xi^n - \xi^{n-1}, \eta^{n-\frac{1}{2}}) - (\rho^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}) + (f(u^{n-\frac{1}{2}})$$

$$- f(u_H^{n-\frac{1}{2}}), \eta^{n-\frac{1}{2}}) - (\varepsilon, \eta^{n-\frac{1}{2}}).$$

By the equation (3) and (4) we obtain the following error estimate equations:

$$\sum_{n=1}^{m}\left(\left\|\eta^{n}\right\|^{2}-\left\|\eta^{n-1}\right\|^{2}\right)+\sum_{n=1}^{m}\Delta t\left\|\theta^{n-\frac{1}{2}}\right\|^{2}$$

$$\leq\sum_{n=1}^{m}(3L+1)\Delta t\left\|\eta^{n-\frac{1}{2}}\right\|^{2}+c\left\|\xi^{n}\right\|^{2}+c\sum_{n=1}^{m}\Delta t\left\|\rho^{n-\frac{1}{2}}\right\|^{2}$$

$$+c\sum_{n=1}^{m}\Delta t\left\|\xi^{n-\frac{1}{2}}\right\|^{2}+\sum_{n=1}^{m}\Delta t\|\varepsilon\|^{2}.$$

Note that $\eta^{0}=0$, 当 $\Delta t\leq(3L+1)^{-1}$ 时, by the lemma Gronwall, we have

$$\left\|\eta^{m}\right\|^{2}+\sum_{n=1}^{m}\Delta t\left\|\theta^{n-\frac{1}{2}}\right\|^{2}\leq c(\Delta t^{4}+H^{6}),$$

By the triangle inequality, we complete the proof.

Lemma 1 Suppose that (u,p) and (u_H,p_H) are the solution of the equation (2) and equation (3), let $u_t\in H^3(\Omega),u\in H^4(\Omega)$, we can obtain that

$$\|u-u_H\|_{0,4}=O(h^{\frac{5}{2}}).$$

Proof : we have known that:

$$\|I_Hu-u_H\|=O(h^3)\left[\int(|u_t|_3^2+|u|_4^2)ds\right]^{\frac{1}{2}},$$

By the lemma 1 and the inverse estimation, we can get

$$\|u-u_H\|_{0,4}\leq\|u-I_Hu\|_{0,4}+\|I_Hu-u_H\|_{0,4}$$

$$\leq c(\|u-I_Hu\|H^{\frac{d}{4}-\frac{d}{2}}+\|I_Hu-u_H\|H^{\frac{d}{4}-\frac{d}{2}})\leq cH^{\frac{5}{2}}(d=2).$$

Theorem 2 under the coarse grid subdivision, $(u_h^n,p_h^n)\in V_h\times W_h$ is the solution of step 1, there is constant and independent of the h, there exists a constant C which is independent of the mesh size, there hold

$$\|u^m-u_h^m\|+\left[\sum_{n=1}^{m}\Delta t\left\|p^{n-\frac{1}{2}}-p_h^{n-\frac{1}{2}}\right\|^2\right]^{\frac{1}{2}}\leq c(\Delta t^2+h^3+H^5)$$

Proof : Let

$$u-u_h=(u-I_hu)+(I_hu-u_h)=\xi+\eta,$$

$$p-p_h=(p-\Pi_hp)+(\Pi_hp-p_h)=\rho+\theta.$$

We can get the error equation:

$$(u_t^{n-\frac{1}{2}}-\frac{1}{\Delta t}(u_h^n-u_h^{n-1}),\eta^{n-\frac{1}{2}})-(p^{n-\frac{1}{2}}-p_h^{n-\frac{1}{2}},\nabla\eta^{n-\frac{1}{2}})$$

$$=(f(u^{n-\frac{1}{2}})-f(u_H^{n-\frac{1}{2}}),\eta^{n-\frac{1}{2}})-(f_u(u_H^{n-\frac{1}{2}})(u_h^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}),\eta^{n-\frac{1}{2}})$$

where $v_h=\eta^{n-\frac{1}{2}},w_h=\nabla\eta^{n-\frac{1}{2}}$.

By the Taylor formula

$$(f(u^{n-\frac{1}{2}})-f(u_H^{n-\frac{1}{2}}),\eta^{n-\frac{1}{2}})-(f_u(u_H^{n-\frac{1}{2}})(u_h^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}),\eta^{n-\frac{1}{2}})$$

$$=(f_u(u_H^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}})+f_{uu}(\alpha^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}})^2$$

$$-f_u(u_H^{n-\frac{1}{2}})(u_h^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}),\eta^{n-\frac{1}{2}})$$

$$=(f_u(u_H^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}})+f_{uu}(\alpha^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}})^2,\eta^{n-\frac{1}{2}}).$$

We get the error equation

$$\frac{1}{\Delta t}(\eta^n-\eta^{n-1},\eta^{n-\frac{1}{2}})+(\theta^{n-\frac{1}{2}},\theta^{n-\frac{1}{2}})$$

$$=-\frac{1}{\Delta t}(\xi^n-\xi^{n-1},\eta^{n-\frac{1}{2}})-(\rho^{n-\frac{1}{2}},\theta^{n-\frac{1}{2}})$$

$$+(f_u(u_H^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}})$$

$$+f_{uu}(\alpha^{n-\frac{1}{2}})(u^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}})^2,\eta^{n-\frac{1}{2}})-(\varepsilon,\eta^{n-\frac{1}{2}}).$$

Similar to the proof of theorem 1, the above equation is multiplied by $2\Delta t$, by the use of the Young inequality, we can obtain that:

$$\sum_{n=1}^{m}\left(\left\|\eta^{n}\right\|^{2}-\left\|\eta^{n-1}\right\|^{2}\right)+\sum_{n=1}^{m}\Delta t\left\|\theta^{n-\frac{1}{2}}\right\|^{2}$$

$$\leq\sum_{n=1}^{m}(F+3)\Delta t\left\|\eta^{n-\frac{1}{2}}\right\|^{2}+c\left\|\xi^{n}\right\|^{2}+c\sum_{n=1}^{m}\Delta t\left\|\rho^{n-\frac{1}{2}}\right\|^{2}$$

$$+c\sum_{n=1}^{m}\Delta t\left\|\xi^{n-\frac{1}{2}}\right\|^{2}+\sum_{n=1}^{m}\Delta t\|\varepsilon\|^{2}+\frac{G}{2}\|u-u_H\|_{0,4}^{4}.$$

note that $\eta^0=0$, let $\Delta t\leq(3+F)^{-1}$, by the lemma Gronwall, we have

$$\left\|\eta^{m}\right\|^{2}+\sum_{n=1}^{m}\Delta t\left\|\theta^{n-\frac{1}{2}}\right\|^{2}\leq c(\Delta t^{4}+h^{6}+H^{10}),$$

By use of the Tayor formula and lemma 1, we can complete the proof.

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