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M/M/1 TWO-PHASE GATED QUEUEING SYSTEM WITH UNRELIABLE SERVER AND STATE DEPENDENT ARRIVALS

S. HANUMANTHA RAO[a*], V. VASANTA KUMAR[b], T. SRINIVASA RAO[b] and B. SRINIVASA KUMAR[b]

[a]VFSTR University, VADLAMUDI – 522213, Dist.: Guntur (A.P.) INDIA
[b]KL University, VADDESWARAM – 522502, Dist.: Guntur (A.P.) INDIA

ABSTRACT

In this paper, we analyze an N-policy, two-phase queueing system where the service station is subject to breakdowns while in operation and repair may delayed due to non-availability of the repair facility. Arrivals follow a Poisson process with rates depending upon the system state namely-vacation, startup, operational and breakdown state. The service is in two essential phases; the first one being batch service to all the customers waiting in the queue and the second one is individual to each of them. The server is turned off each time the system empties. As and when the total number of customers in the system reaches the threshold $N(N \geq 1)$, the server is turned on and requires preparatory time before starting the batch service. The customers who arrive during batch service are not allowed to join the batch, which is in service, but are bunched together and are served along with the other arrivals during the next visit of the server to the batch queue. Startup times, uninterrupted service times, length of each delay period and repair period follows exponential distribution. Closed form expressions for the mean system size at various states of the server are derived. Effect of the system parameters on the optimal threshold N is studied through numerical examples.

Key words: Two-phase, Vacation, Breakdowns, N-policy, Delayed repair, State dependent arrival rates.

INTRODUCTION

In many real-life queuing systems like communication systems, manufacturing systems, and computer networks, the server is subject to unpredictable breakdowns and can be repaired. The performance of such systems may be affected by the breakdowns of the service station and delay in repair due to non-availability of the repairman or of the

*Author for correspondence; E-mail: sama.hanumantharao@gmail.com, vvkumar@kluniversity.in, tsr_2505@kluniversity.in, sk_bhavirisetty@kluniversity.in

apparatus needed for repairs. Therefore, it is necessary to see how the breakdowns affect the server's level of performance. The arrival of customers may depend on the state of the system. Gray et al.[1] analyzed a multiple-vacation queueing model where the service station is subject to breakdowns while in operation and the arrival rates depend upon the state of the system. Hanumantha Rao et al.[2] presented the optimal operating policy of an N-policy two phase M/M/1 queueing model with unreliable server, server startup and state dependent arrival rates. Vasanta Kumar et al.[3] studied optimal strategy analysis of an N-policy two-phase $M/E_k/1$ queueing system with server startup, breakdowns and gating.

Present study is aimed to analyze the economic behaviour of an N-policy M/M/1 gated queue with service in two phases, state dependent arrival rates and the server is typically subject to unpredictable breakdowns and delay in repair.

Mathematical model

The following assumptions and notations are used to study the steady state behaviour of the model under consideration.

The service is in two phases, the first one being batch service to all waiting customers in the queue and the second one is individual to each of them. The uninterrupted batch and individual service times are of exponential lengths. The server goes on vacation at the instant when the queue becomes empty and continues to take vacation until N customers accumulate. The server needs a startup time for preparatory work, which is of exponential length. The service mechanism breakdowns occur only during active service and repair will not take place immediately due to non-availability of the repair facility. Hence, there will be delay in repair. Server breakdowns occur at a poisson rate. The delay times and repair times are of exponential length. The arrival processes during vacation, startup, active service, and breakdown are poisson with different arrival rates. All inter-arrival, vacation, startup, service, inter-breakdown, delay and repair times are independent of each other. By gating, we mean that the customers who arrive during batch service are not allowed to enter the batch, which is already in service, but are bunched together and served along with other arrivals during the next visit of the server to the batch queue.

Notations:

λ_1 : Arrival rate during vacation and startup

λ_2 : Arrival rate during batch or individual service

λ_3 : Arrival rate during vacation

θ : Vacation rate

β : Batch service rate

μ : Individual service rate

ξ : Breakdown rate

η : Delay rate

α : Repair rate

In order to study the steady state behavior of the system, the following steady state probabilities are defined.

V (i,0) = The probability that there are i customers in the batch queue when the server is on vacation, I = 0,1, 2,...N-1.

Q (i,0) = The probability that there are i customers in the batch queue while the server is in pre-service, where i = N , N+1 , N+2,…

B (i,0) = The probability that there i customers in the batch, which is in batch service, i = 1, 2, 3…

P (i, j) = The probability that there are i customers in the batch queue and j customers in the individual queue while the server is in individual service, i = 0, 1, 2,… and j = 1, 2, 3…

P_d (i, j) = The probability that there are i customers in the batch queue and j customers in the individual queue while the server is in individual service, but found to be broken down and waiting for repair , i = 0,1,2…, and j = 1,2,3….

P_r (i, j) = The probability that there are i customers in the batch queue and j customers in the individual queue while the server is in individual service, but the server is under repair, i = 0,1,2…, and j = 1,2,3….

The steady state results

The steady state equations satisfied by the system size probabilities are as follows:

$$\lambda_1 V(0,0) = \mu P(0,1) \quad \ldots(1)$$

$$\lambda_1 V(i,0) = \lambda_1 V(i-1, 0),\ 1 \le i \le N-1 \quad \ldots(2)$$

$$(\lambda_1 + \theta)\, Q(N, 0) = \lambda_1 V(N-1, 0) \quad \ldots(3)$$

$$(\lambda_1 + \theta)\, Q(i, 0) = \lambda_1 Q(i-1, 0),\ i > N \quad \ldots(4)$$

$$\beta\, B(i,0) = \mu P(i,\, 1),\ 1 \le i \le N - 1 \quad \ldots(5)$$

$$\beta\, B(i,0) = \mu P(i,\, 1) + \theta Q\,(i,\, 0),\ i > N \quad \ldots(6)$$

$$(\lambda_2 + \xi + \mu)P(0,\, j) = \mu P(0,\, j + 1) + \beta\Pi_0\, B(j,\, 0)P + \alpha P_r(0,\, j),\ j \ge 1 \quad \ldots(7)$$

$$(\lambda_2 + \xi + \mu)P(i,\, j) = \mu P(i,\, j + 1) + \beta\Pi_i\, B(j,\, 0) + \lambda_2\, P(i - 1,\, j) + \alpha P_r(0,\, j),\ i \ge 1,\ j \ge 1 \quad \ldots(8)$$

$$(\lambda_3 + \eta)\, P_d(0,\, j) = \xi P(0,\, j),\ j \ge 1 \quad \ldots(9)$$

$$(\lambda_3 + \eta)\, P_d(i,\, j) = \xi P(i,\, j) + \lambda_3 P_d\,(i - 1,\, j),\ i \ge 1,\ j \ge 1 \quad \ldots(10)$$

$$(\lambda_3 + \alpha)\, P_r(0,\, j) = \eta P_d\,(0,\, j),\ j \ge 1 \quad \ldots(11)$$

$$(\lambda_3 + \alpha)\, P_r(i,\, j) = \eta P_d\,(i,\, j) + \lambda_3 P_r\,(i - 1,\, j),\ i \ge 1,\ j \ge 1 \quad \ldots(12)$$

where $\Pi_i = \dfrac{\lambda_2^i \beta}{(\lambda_2 + \beta)^{i+1}}$, π_i is the probability that there are i arrivals during batch service. Define $\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i, |z| \le 1$.

We define the following generating functions to solve the steady state equations

$$F_v(z) = \sum_{i=0}^{N-1} V(i,0)z^i\ ,\ F_q(z) = \sum_{i=N}^{\infty} Q(i,0)z^i\ ,\ F_b(z) = \sum_{i=1}^{\infty} B(i,0)z^i,$$

$$F_d(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_d(i,j)z^i y^j\ ,\ F_r(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_r(i,j)z^i y^j\ ,$$

$$F_p(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P(i,j)z^i y^j,\quad R_j(z) = \sum_{i=0}^{\infty} P(i,j)z^i\ ,\ S_j(z) = \sum_{i=0}^{\infty} P_d(i,j)z^i,$$

$$T_j(z) = \sum_{i=0}^{\infty} P_r(i,j)z^i, |z| \le 1 \text{ and } |y| \le 1$$

Multiply equation (2) by z^i, sum over from 1 to N-1 and add equation (1). Then we have –

$$F_v(z) = \frac{(1-z^N)}{(1-z)} V(0,0) \quad \ldots(13)$$

Multiply equation (4) by z^i, sum over i from (N+1) to ∞ and add z^N times equation (3). Then we have –

$$F_q(z) = \frac{\lambda_1 z^N V(0,0)}{(\lambda_1(1-z)+\theta)} \quad ...(14)$$

Multiply equations (5) and (6) by z^i, sum over i from 1 to ∞. Then we have –

$$\beta F_b(z) = \mu R_1(z) + \theta F_q(z) - \lambda_1 V(0,0) \quad ...(15)$$

Multiply equation (8) by z^i, sum over from 1 to ∞ and add equation (7). Then we have

$$[\lambda_2(1-z) + \xi + \mu] R_j(z) = \mu R_{j+1}(z) + \beta B(j,0)\Pi(z) + \alpha T_j(z) \quad ...(16)$$

Multiply this equation by y^j and sum over j from 1 to ∞. Then we have –

$$[\lambda_2 y(1-z) + \xi y + \mu(y-1)] F_p(z,y) = \beta y\, \Pi(z) F_q(y) - \mu y R_1(z) + \alpha y F_r(z,y) \quad ...(17)$$

Similarly from the equations (10) to (13), we get –

$$[\lambda_3(1-z) + \eta] F_d(z,y) = \xi\, F_p(z,y) \quad ...(18)$$

$$[\lambda_3(1-z) + \alpha] F_r(z,y) = \eta F_d(z,y) \quad ...(19)$$

Substitute the value of $F_r(z,y)$ in equation (17). Then we have –

$$\left[\lambda_2 y(1-z) + \xi y + \mu(y-1) - \frac{\alpha\xi\eta y}{(\lambda_3(1-z)+\alpha)(\lambda_3(1-z)+\eta)}\right] F_p(z,y)$$

$$= -\mu y R_1(z) + \beta y \Pi(z) F_b(y) \quad ...(20)$$

Put y = z and substitute the value of $F_b(z)$ from equation (15) in equation (20) and cancel the common factor (z-1) on both sides. Then –

$$\frac{(\mu - \lambda_2 z)(\lambda_3(1-z)+\alpha)(\lambda_3(1-z)+\eta) + \xi\lambda_3^2 z(1-z) - \xi\lambda_3(\alpha+\eta)z}{(\lambda_3(1-z)+\alpha)(\lambda_3(1-z)+\eta)} F_p(z,z)$$

$$= \mu z R_1(z)\frac{(\Pi(z)-1)}{z-1} + \frac{\lambda_1 \Pi(z) z V(0,0)\left(\theta\left(\frac{z^N-1}{z-1}\right) + \lambda_1\right)}{(\lambda_1(1-z)+\theta)} \quad ...(21)$$

This can be written as –

$$\Phi(z)F_p(z,z) = \left[\mu z R_1(z)\left(\frac{(\Pi(z)-1)}{z-1}\right) + \frac{\lambda_1 \Pi(z) z V(0,0)\left(\theta\left(\frac{z^N-1}{z-1}\right)+\lambda_1\right)}{(\lambda_1(1-z)+\theta)}\right]$$

$$(\lambda_3(1-z)+\delta)(\lambda_3(1-z)+\eta) \quad \ldots(22)$$

Where

$$\Phi(z) = \left[-\lambda_2\lambda_3^2 z^3 + \left(2\lambda_2\lambda_3^2 + \lambda_2\lambda_3(\eta+\alpha) + \mu\lambda_3^2 + \lambda_3^2\xi\right)z^2 + \left(-\lambda_2\lambda_3^2 - \lambda_2\lambda_3(\eta+\alpha) - \lambda_2\eta\alpha - 2\mu\lambda_3^2 - \lambda_3\mu(\eta+\alpha) - \lambda_3^2\xi - \lambda_3\xi(\eta+\alpha)\right)z + \left(\mu\lambda_3^2 + \lambda_3\mu(\eta+\alpha) + \mu\eta\alpha\right)\right] \quad \ldots(23)$$

Put z = 1 and y = 1 in equations (13), (14), (19), (21), and (22). Then –

$$F_v(1) = NV(0,0) \quad \ldots(24)$$

$$F_q(1) = \frac{\lambda_1 V(0,0)}{\theta} \quad \ldots(25)$$

$$F_b(1) = \frac{\mu R_1(1)}{\beta} \quad \ldots(26)$$

$$F_p(1,1) = \alpha\eta\left[\mu R_1(1)\Pi'(1) + \lambda_1 V(0,0)\frac{(\lambda_1+N\theta)}{\theta}\right]\frac{1}{\Phi(1)} \quad \ldots(27)$$

$$F_d(1,1) = \frac{\xi F_p(1,1)}{\eta} \quad \ldots(28)$$

and

$$F_r(1,1) = \frac{\xi F_p(1,1)}{\alpha}. \quad \ldots(19)$$

Where $\Pi'(1) = \lambda_2/\beta$.

The probability that the server is in vacation or startup is given by –

$$F_v(1) + F_q(1) = 1 - \frac{\lambda_2}{\beta} - \frac{\lambda_2}{\mu} - \frac{\lambda_3}{\mu}\left(\frac{\xi}{\eta} + \frac{\xi}{\alpha}\right)$$

This gives –

$$\left(\frac{\lambda_1 + N\theta}{\theta}\right)V(0,0) = 1 - \frac{\lambda_2}{\beta} - \frac{\lambda_3}{\mu}\left(\frac{\lambda_2}{\lambda_3} + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right) = 1 - \rho$$

Where $$\rho = \frac{\lambda_2}{\beta} + \frac{\lambda_3}{\mu}\left(\frac{\lambda_2}{\lambda_3} + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right) \quad \ldots(30)$$

Hence the stationary queue length distribution exists if $\rho < 1$.

The generating function of the queue length distribution is given by –

$$\mathrm{F}(\mathrm{z},\mathrm{z}) = F_v(\mathrm{z}) + F_q(\mathrm{z}) + F_{\mathrm{b}}(\mathrm{z}) + \mathrm{F}_p(\mathrm{z},\mathrm{z}) + F_d(\mathrm{z},\mathrm{z}) + F_{\mathrm{r}}(\mathrm{z},\mathrm{z}) \quad \ldots(31)$$

The normalizing condition is F(1,1) = 1.

This condition gives –

$$\frac{\mu}{\beta}\left[1 + \frac{\lambda_2\eta\alpha}{\Phi(1)}\left(1 + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right)\right] R_1(1) = \rho - \frac{\eta\alpha\lambda_1}{\Phi(1)}\left(1 + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right)(1-\rho) \quad \ldots(32)$$

Using the condition$\lim_{\mathrm{y}\to 1} F_p(1,\mathrm{y}) = \lim_{\mathrm{z}\to 1} F_p(\mathrm{z},1)$, we obtain –

$$\left[1 - \frac{\lambda_3}{\mu}\left(\frac{\lambda_2}{\lambda_3} + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right)\right] R_1'(1) = \frac{\lambda_1\lambda_3}{\mu^2}(1-\rho) \quad \ldots(33)$$

We now determine the roots of Φ(z) =0 for positive λ_3. Referring to (23), Φ(z) is cubic equation. Φ(z) has three changes of sign and Φ(-z) has no change of sign. By Descarterule of signs the equation Φ(z) = 0 has three positive real roots. In order that steady state queue length distribution to exist, all the three roots of the equation Φ(z) = 0 must be greater than one. Since the coefficient of z^3 in Φ(z) is negative, the roots of Φ(z) = 0 will be greater than 1, if and only if $\Phi(1) > 0$, $\Phi'(1) < 0$ and $\Phi''(1) > 0$. Since $\Phi(1) = (\mu - \lambda_2)\eta\alpha - \lambda_3\xi(\eta + \alpha)$,we must assume that$\mu\eta\alpha > \lambda_2\eta\alpha + \lambda_3\xi(\eta + \alpha)$.

This gives –

$$\frac{\lambda_2}{\mu} + \frac{\lambda_3\xi}{\mu}\left(\frac{1}{\eta} + \frac{1}{\alpha}\right) < 1 \quad \ldots(34)$$

Now (30) and (34) implies that μ is greater than β, λ_2 and λ_3. Hence if (34) holds, then –

$$\Phi'(1) = \lambda_3(\lambda_2 - \mu)(\eta + \alpha) - \lambda_2\eta\alpha - \lambda_3\xi(\eta + \alpha) < 0$$

and $$\Phi''(1) = 2\lambda_3^2(\mu - \lambda_2) + 2\lambda_2\lambda_3(\eta + \alpha) + 2\lambda_3^2\alpha > 0$$

Thus, if we assume that (34) holds, then the roots z_1, z_2 and z_3 of $\Phi(z) = 0$ will be greater than 1.Under the condition (34), choose λ_2, λ_3 and β such that $0 < V(0,0) < 1$.

Let $k_1 = \frac{1}{z_1}, k_2 = \frac{1}{z_2}$ and $k_3 = \frac{1}{z_3}$

Then $\Phi(z) = \mu(\lambda_3^2 + \lambda_3(\eta + \alpha) + \eta\alpha)(1 - k_1 z)(1 - k_2 z)(1 - k_3 z)$.

Now, from (22).

$$\mu\left(\lambda_3^2 + \lambda_3(\eta + \alpha) + \eta\alpha\right)(1 - k_1 z)(1 - k_2 z)(1 - k_3 z)F_p(z, z)$$

$$= \left[\mu z R_1(z)\frac{(\Pi(z) - 1)}{z - 1} + \frac{\lambda_1 \Pi(z) z V(0,0)\left(\theta\left(\frac{z^{N} - 1}{z - 1}\right) + \lambda_1\right)}{(\lambda_1(1 - z) + \theta)}\right](\lambda_3(1 - z) + \eta)(\lambda_3(1 - z) + \alpha) \quad \ldots(35)$$

Expected number of customers in the system

Using the probability generating functions expected number of customers in the system at different states are presented in this section. Let $L_v, L_q, L_b, L_p, L_d,$ and $L_r,$ be the expected number of customers in the system when the server is in vacation, in startup, in batch service, in individual service, waiting for repair during individual service and under repair during individual service states, respectively. Then –

$$L_v = \sum_{i=1}^{N-1} i\, V(0,0) = F_v'(1) = \frac{N(N-1)}{2} V(0,0), \quad \ldots(36)$$

$$L_q = \sum_{i=1}^{\infty} i\, Q(i, 0) = F_q'(1) = \frac{\lambda_1(\lambda_1 + N\theta)}{\theta^2} V(0,0), \quad \ldots(37)$$

$$L_b = \sum_{i=1}^{N-1} i\, B(i, 0) = F_b'(1) = \frac{\mu}{\beta} R_1'(1) + \frac{\lambda_1(\lambda_1 + N\theta)}{\theta} V(0,0) \quad \ldots(38)$$

$$L_p = \sum_{i=1}^{\infty} i\, P(i, j) P_{3,i,0} = F_p'(1,1) = \quad \ldots(39)$$

$$= \left[\frac{k_1}{1 - k_1} + \frac{k_2}{1 - k_2} + \frac{k_3}{1 - k_3}\right] F_p(1,1) + \frac{\eta\alpha}{\Phi(1)}\left[\frac{\mu R_1(1)\pi''(1)}{2} + \mu\pi'(1)(R_1(1) + R_1'(1))\right]$$

$$+ \frac{\lambda_1 \eta\alpha\, V(0,0)}{\theta\Phi(1)}\left[\left(1 + \Pi'(1)\right)(\lambda_1 + N\theta) + \frac{N(N-1)}{2}\right] + \lambda_1^2 \eta\alpha \frac{(\lambda_1 + N\theta)}{\theta^2 \Phi(1)} V(0,0)$$

$$-\lambda_3\left[\mu R_1(1)\Pi'(1)+\frac{\lambda_1(\lambda_1+N\theta)}{\theta}\mathrm{V}(0,0)\right]\frac{(\gamma+\delta)}{\Phi(1)}$$

$$\mathrm{L_d}=\sum_{\mathrm{i=1}}^{\mathrm{N-1}}\mathrm{i}\ \mathrm{P}_d(\mathrm{i,j})=F_d'(1,1)=\frac{\lambda_3\xi}{\eta^2}F_p(1,1)+\frac{\xi}{\eta}F_p'(1,1) \quad \ldots(40)$$

$$\mathrm{L}_r=\sum_{\mathrm{i=0}}^{\infty}\sum_{\mathrm{j=1}}^{\infty}(\mathrm{i+j})\ \mathrm{P}_r(\mathrm{i,j})=F_r'(1,1)=\frac{\lambda_3}{\alpha}\left(\frac{\xi}{\eta}+\frac{\xi}{\alpha}\right)F_p(1,1)+\frac{\xi}{\alpha}F_p'(1,1) \quad \ldots(41)$$

The expected number of customers in the system is given by –

$$\mathrm{L(N)}=\mathrm{L_v}+\mathrm{L_q}+\mathrm{L_b}+\mathrm{L}_p+\mathrm{L}_d+\mathrm{L}_r \quad \ldots(42)$$

Where $\Pi''(1)=\frac{2\lambda_2^2}{\beta^2}$ and $R_1(1)$ and $R_1'(1)$ are given by (32) and (33), respectively.

Some other system characteristics

$\mathrm{W_v}$, $\mathrm{W_q}$, $\mathrm{W_b}$, $\mathrm{W_p}$, $\mathrm{W_d}$ and $\mathrm{W_r}$ denote the expected length of vacation period, startup period, batch service period, individual service period, delay period during individual service and waiting period for repair during individual service respectively, then the expected length of a cycle is given by

$$\mathrm{W_c}=\mathrm{W_v}+W_q+W_{\mathrm{b}}+W_p+W_d+W_r$$

The long run fractions of time the server is in different states are, respectively.

$$\frac{W_{\mathrm{v}}}{\mathrm{W_c}}=F_{\mathrm{v}}(1)=\mathrm{NV}(0,0) \quad \ldots(43)$$

$$\frac{W_{\mathrm{q}}}{\mathrm{W_c}}=F_q(1)=\frac{\lambda_1\mathrm{V}(0,0)}{\theta} \quad \ldots(44)$$

$$\frac{W_{\mathrm{b}}}{\mathrm{W_c}}=F_{\mathrm{b}}(1)=\frac{\mu\mathrm{R}_1(1)}{\beta} \quad \ldots(45)$$

$$\frac{W_p}{\mathrm{W_c}}=F_p(1,1)=\frac{\eta\alpha}{\Phi(1)}\left[\lambda_2+\lambda_3\left(\frac{\xi}{\eta}+\frac{\xi}{\alpha}\right)+\frac{\lambda_1(\lambda_1+\mathrm{N}\theta)}{\theta}\mathrm{V}(0,0)\right] \quad \ldots(46)$$

$$\frac{\mathrm{W_d}}{\mathrm{W_c}}=F_d(1,1)=\frac{\xi}{\eta}F_p(1,1) \quad \ldots(47)$$

and

$$\frac{\mathrm{W_r}}{\mathrm{W_c}}=F_r(1,1)=\frac{\xi}{\alpha}F_{\mathrm{p}}(1,1) \quad \ldots(48)$$

The expected length of vacation period $W_V = \frac{N}{\lambda_1}$. Substituting this in equation (41),

$$W_C = \frac{1}{\lambda_1 V(0,0)}$$

Optimal control policy

In this section we determine the optimal value of N, which minimizes the long run average cost for the N-policy M/M/1 gated queue with server break downs and delay in repair. The following linear cost structure is considered.

A(N) be the average cost per unit of time , then

$$A(N) = C_h L(N) + C_0\left(\frac{W_b}{W_c} + \frac{W_p}{W_c}\right) + C_m\left(\frac{W_q}{W_c}\right) + C_b\left(\frac{W_d}{W_c} + \frac{W_r}{W_c}\right)$$
$$+ C_s\left(\frac{1}{W_c}\right) - C_r\left(\frac{W_v}{W_c}\right) \quad \ldots(49)$$

where $C_h \equiv$ Holding cost per unit time for each customer present in the system,

$C_0 \equiv$ Cost per unit time for keeping the server on and in operation,

$C_m \equiv$ Startup cost per unit time,

$C_s \equiv$ Setup cost per cycle,

$C_b \equiv$ Break down cost per unit time for the unavailable server, and

$C_r \equiv$ Reward per unit time as the server is doing secondary work during vacation.

From (45) to (48), it is observed that $\frac{W_b}{W_c}, \frac{W_p}{W_c}, \frac{W_d}{W_c}$ and $\frac{W_r}{W_c}$ are independent of the decision variable N. Hence for determination of the optimal operating N-policy, minimizing A(N) in (49) is equivalent to minimizing.

$$A_1(N) = C_h L(N) + C_m\left(\frac{W_q}{W_c}\right) + C_s\left(\frac{1}{W_c}\right) - C_r\left(\frac{W_b}{W_c}\right) \quad \ldots(50)$$

Differentiating $A_1(N)$ with respect to N and setting the result to 0, we obtain the optimal threshold N^* of N. Hence

$$N^* = \sqrt{\sigma(\sigma+1) + \frac{2\sigma}{BC_h}(C_m + \theta C_s + C_r)} - \sigma$$

where $\sigma = \frac{\lambda_1}{\theta}$ (mean number of arrivals during startup time) and $B = 1 + \frac{\lambda_1 \alpha}{\theta \Phi(1)}\left(1 + \frac{\xi}{\eta} + \frac{\xi}{\alpha}\right)$.

Sensitivity analysis

It is observed from Table 1 that (i) N^* is convex with increase in the values of λ_1, insensitive with increase in λ_2 and λ_3, and (ii) $L(N^*)$ and $T(N^*)$ increase with increase in the values of λ_1, λ_2, and λ_3.

Table 1: The optimal N^*, $L(N^*)$ and minimum expected cost $T(N^*)$ by varying (λ_1, λ_2, λ_3,)

($\mu = 5$, $\beta = 10.0$, $\theta = 0.7$, $\xi = 0.5$, $\alpha = 4$, $\eta = 5$, $C_h = 25$, $C_m = 50$, $C_b = 30$, $C_s = 500$, $C_o = 40$, $C_r = 20$)

$\lambda_2 = 1.0, \lambda_3 = 5$				$\lambda_1 = 0.5, \lambda_3 = 1.5$				$\lambda_1 = 0.5, \lambda_2 = 1.0$			
λ_1	N^*	$L(N^*)$	$T(N^*)$	λ_2	N^*	$L(N^*)$	$T(N^*)$	λ_3	N^*	$L(N^*)$	$T(N^*)$
0.5	4	4.69	149.59	1.1	4	5.23	162.45	2.5	4	5.00	157.05
1.0	6	6.84	210.47	1.3	4	6.66	196.36	5.0	4	6.09	183.20
3.0	8	11.56	343.81	1.4	4	7.58	218.31	6.5	4	7.08	207.00
6.0	9	16.35	462.05	1.5	4	8.66	244.32	8.0	4	8.50	241.35
9.5	8	20.56	564.59	1.6	4	9.93	275.06	8.5	4	9.12	256.26
17.5	7	30.41	799.29	1.7	4	11.43	311.33	9.5	4	10.65	293.66

From Table 2 it can be seen that (i) with increase in the values of μ, β and θ, N^* is insensitive, (ii) $L(N^*)$ increases with increase in the value of μ and decreases with increase in β and θ, and iii) $T(N^*)$ increases with increase in the values of μ and β, and decrease with increase in θ.

It can be seen from Table 3 that (i) N^* and $L(N^*)$ are insensitive with increase in the values of C_b and shows increasing trend with increase in C_m and C_s, and (ii) $T(N^*)$ is insensitive with increase in the values of C_b and increases with increase in C_s and C_m.

Table 2: The optimal N^*, $L(N^*)$ and minimum expected cost $T(N^*)$ by varying (μ, β, θ)

($\lambda_1 = 0.5, \lambda_2 = 1.0, \lambda_3 = 1.5$, $\alpha = 4$, $\xi = 0.5$, $\eta = 5$, $C_h = 25$, $C_m = 50$, $C_b = 30$, $C_s = 500$, $C_o = 40$, $C_r = 20$).

$\beta = 10.0, \theta = 0.7$				$\mu = 5.0, \theta = 0.7$				$\mu = 5.0, \beta = 10.0$			
μ	N^*	$L(N^*)$	$T(N^*)$	β	N^*	$L(N^*)$	$T(N^*)$	Θ	N^*	$L(N^*)$	$T(N^*)$
5	4	4.69	149.59	12	4	4.67	150.01	0.7	4	4.69	149.59
10	4	4.35	144.90	14	4	4.66	150.24	1.2	4	4.41	142.29
15	4	4.50	149.69	16	4	4.65	150.39	1.7	4	4.29	139.25
20	4	4.73	156.03	18	4	4.63	150.49	2.2	4	4.23	137.58
25	4	4.99	162.89	20	4	4.63	150.55	2.7	4	4.19	136.52
30	4	5.27	169.99	22	4	4.62	150.60	3.2	4	4.16	135.80

Table 3: The optimal N^*, $L(N^*)$ and minimum expected cost $T(N^*)$ by varying (C_b, C_s, C_m)

($\lambda_1 = 0.5, \lambda_2 = 1.0, \lambda_3 = 1.5$, μ=5.0, β=10.0, θ=0.7, ξ=0.5, α=4, η=5, C_h=25, C_o=40, C_r=20)

C_s =500, C_m=50				C_b=30, C_m =50				C_s=500, C_b=30			
C_b	N^*	$L(N^*)$	$T(N^*)$	C_s	N^*	$L(N^*)$	$T(N^*)$	C_m	N^*	$L(N^*)$	$T(N^*)$
30	4	4.69	149.59	550	4	4.69	153.53	100	5	5.13	156.94
50	4	4.69	149.59	600	5	5.13	158.79	200	5	5.13	166.22
70	4	4.69	149.59	900	6	5.57	179.74	300	5	5.13	175.50
90	4	4.69	149.59	1300	7	6.02	203.04	400	6	5.57	185.27
110	4	4.69	149.59	1700	8	6.47	223.64	500	6	5.57	193.16
130	4	4.69	149.59	2000	8	6.47	236.42	600	7	6.02	202.36

From Table 4 we observe that (i) $N^*$and $L(N^*)$decrease with increase in the values of C_h and insensitive with increase in C_o and C_r, and (ii) $T(N^*)$ increases with increase in C_h, insensitive with C_o and decreases with C_r.

Table 4: The optimal N^*, $L(N^*)$ and minimum expected cost $T(N^*)$ by varying (C_h, C_0, C_r)

($\lambda_1 = 0.5, \lambda_2 = 1.0, \lambda_3 = 1.5$, $\xi = 0.5$, $\mu = 5.0$, $\beta = 10.0$, $\theta = 0.7$, $\alpha = 4$, $\eta = 5$, $C_m = 50$, $C_b = 30$, $C_s = 500$)

C_o=40, C_r=20				C_h=25, C_r=20				C_h= 25, C_o=40			
C_h	N^*	$L(N^*)$	$T(N^*)$	C_o	N^*	$L(N^*)$	$T(N^*)$	C_r	N^*	$L(N^*)$	$T(N^*)$
25	4	4.69	149.59	50	4	4.69	149.59	50	4	4.69	130.69
45	3	4.26	236.82	100	4	4.69	149.59	60	4	4.69	130.69
55	3	4.26	279.42	150	4	4.69	149.59	70	4	5.13	119.82
65	2	3.86	317.85	200	4	4.69	149.59	80	4	5.13	113.32
75	2	3.86	365.49	250	4	4.69	149.59	90	4	5.13	106.83
165	1	3.52	695.41	300	4	4.69	149.59	100	4	5.13	100.33

CONCLUSION

An N-policy two-phase M/M/1 gated queueing of an unreliable server with pre-service work, delay repair and state dependent arrival rates is studied. Some of the system performance measures are obtained. Sensitivity of the optimal threshold N, expected system length and average cost with changes in the system parameters and cost elements is also studied.

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