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AN EVALUATION OF CONDENSATE FRACTION (N_0/N) AS A FUNCTION OF (T/T_C^0) FOR INTERACTING AND NON-INTERACTING MODELS WITH TWO DIFFERENT TRAPS

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ABSTRACT

Using theoretical formalism of F. Dalfovo et. al., condensate fraction of BEC (N_o/N) as a function of (T/T_c^o) were evaluated using non-interacting and interacting models for two different traps and compared with scaling parameter η. Our results indicate that two very different configuration can give rise to the same thermodynamic behavior of the condensate corresponding to the scaling parameter η.

Key words: Bose-Einstein condensation, Condensate, Scaling parameter, Thermodynamic behavior, Interacting models.

INTRODUCTION

BEC was observed in remarkable series of experiments on vapor of rubidium, sodium[2] and lithium[3]. In all these experiments, atoms were confined in magnetic traps and cooled down to extremely low temperature, of the order of fraction of micro Kelvin. The first evidence for condensation emerged from time of flight measurements. The atoms were left to expand· by switching off the confining trap and then imaged with optical method. A sharp peak in the velocity distribution was then observed below a certain critical temperature providing a clear signature of BEC.

One of-the most relevant features of these trapped Bose gases is that they are inhomogeneous and finite sized systems. The number of atom ranges typically from few thousands to several million. In most cases, the confing traps are well approximated by harmonic potential. The trapping frequency ω_{ho} provides a characteristic length scale of the system as -

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$$a_{ho} = \left(\hbar / m\omega_{ho}\right)^{1/2} \qquad \ldots(1)$$

which is in the order of few microns. There is a major difference between BEC of uniform Bose gas (like super fluid helium) and BEC of trapped gases. In the case of super fluid helium, the effect of inhomogeneity takes place on a microscopic scale fixed by the inter-atomic distance whereas in the case of trapped Bose gas, the size of the system is enlarged as an effect of repulsive two body forces. Therefore, the trapped gases can become almost macroscopic objects, directly measureable with optical methods. The fact that these gases are highly inhomogeneous has several important consequences. First BEC shown up not in the momentum space, as happens in super fluid helium, but also in coordinate space. This double possibility of investigating the effect of condensation is very interesting from both; the theoretical and experimental point of view. This provides novel methods of investigation for relevant quantities like the temperature dependence of the condensate, energy and density distribution, interference phenomenon, frequencies of collective excitations and so on[4].

Mathematical formulae used in the evaluation

The condensate of non-interacting Bosons

One starts with taking the confining potential in a quadratic form

$$V_{ext}(r) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_x^2 y^2 + \omega_x^2 z^2\right) \qquad \ldots(2)$$

Here, one neglects the atom-atom interactions. The many-body Hamiltonian is the sum of single-particle Hamiltonians, whose Eigen values have the form.

$$\varepsilon_{n_z n_y n_z} = (n_x + \frac{1}{2})\hbar\omega_x + (n_y + \frac{1}{2})\hbar\omega_y + (n_z + \frac{1}{2})\hbar\omega_z \qquad \ldots(3)$$

Where $\{n_x, n_y, n_z\}$ are non-negative integers. The ground state $\phi\ (r_1 \ldots r_N)$ of N non-interacting Bosons confined by the potential (2) is obtained by putting all the particles in the lowest single-particle state $\{n_x = n_y = n_z = 0)$, namely $\phi\ (r_1 \ldots r_N) = \Pi_i\, \varphi_o$ where $\varphi_o\ (r)$ is given by -

$$\varphi_o\ (r) = \left(\frac{m\omega_{ho}}{\pi\hbar}\right)^{3/4} \exp\left[-(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)\right] \qquad \ldots(4)$$

One has introduced the geometric average of the oscillator frequencies :

$$\omega_{ho} = (\omega_x \omega_y \omega_z)^{1/3} \quad \ldots(5)$$

The density distribution then becomes $n(\bar{r}) = N \left| \phi_o(\bar{r}) \right|^2$ and its value grows with N. The size of the cloud is independent of N and is fixed by the harmonic oscillator length :

$$a_{ho} = \left(\frac{\hbar}{m\omega_{ho}} \right)^{1/2} \quad \ldots(6)$$

which corresponds to the average width of the Gaussian in Eq. (3).

Trapped Bosons at finite temperature (Thermodynamic limit)

At temperature T, the total number of particle is given, in the grand canonical ensemble, by the sum -

$$N = \sum_{n_x, n_y, n_z} \left\{ \exp\left[\beta (\varepsilon_{n_x n_y n_z} - \mu) \right] - 1 \right\}^{-1} \quad \ldots(7)$$

while the total energy is given by -

$$N = \sum_{n_x, n_y, n_z} \varepsilon_{n_x n_y n_z} \left\{ \exp\left[\beta (\varepsilon_{n_x n_y n_z} - \mu) \right] - 1 \right\}^{-1} \quad \ldots(8)$$

Where μ is the chemical potential and $\beta = (K_\beta T)^{-1}$. Below a given temperature, the population of the lowest state becomes macroscopic and this corresponds to the onset of Bose-Einstein condensation.

The chemical potential is given by -

$$\mu \to \mu_c = \frac{3}{2} \hbar\omega \quad \ldots(9)$$

Where $\omega = (\omega_x + \omega_y + \omega_z)/3$ is the arithmetic average of the trapping frequencies. Inserting this value in the rest of the of the sum, one can write

$$N - N_O = \sum_{n_x, n_y, n_z \neq o} \frac{1}{\exp\left[\beta\hbar(\omega_x n_x + \omega_x n_y + \omega_z n_z) \right] - 1} \quad \ldots(10)$$

In order to evaluate this sum explicitly, one usually assumes that the level spacing becomes smaller and smaller, when $N \to \infty$, so that the sum can be replaced by an integral -

$$N - N_o = \int_o^\infty \frac{dn_x\, dn_y\, dn_z}{\exp\left[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)\right] - 1} \quad \ldots(11)$$

This assumption corresponds to a semi-classical description of the excited states. Its validity implies that the relevant excitation energies, contributing to the sum (10) are much larger than the level spacing fixed by the oscillator frequencies. The accuracy of the semi-classical approximation (11) is expected to be good, if the number of trapped atoms is large and $K_\beta T \gg \hbar\omega_{ho}$.

The integral (11) can be easily calculated by changing variables $\beta\hbar\omega_x n_x = \bar{n}_x$ etc.). One finds -

$$N - N_o = \xi(3)\left(\frac{K_B T}{\hbar\omega_{ho}}\right)^3 \quad \ldots(12)$$

Where $\xi(n)$ is the Riemann ξ function and ω_{ho} is the geometric average (5). From this result, one can also obtain the transition temperature for Bose-Einstein condensation by imposing that $N_o \to 0$ at the transition, one gets -

$$K_B T = \hbar\omega_{ho}\left(\frac{N}{\xi(3)}\right)^{1/3} = 0.94\, \hbar\omega_{ho} N^{1/3} \quad \ldots(13)$$

For temperatures higher than T_c^o, the chemical potential is less than μ_c and becomes N dependent, while the population of the lowest state is of the order 1 instead of N. The proper thermodynamic limit for these systems is obtained by letting $N \to \infty$ and $\omega_{ho} \to \infty$, while keeping the product $N\omega^3_{ho}$ constant. With this definition, the transition temperature (13) is well defined in the thermodynamic limit. Inserting the above expression for T_c^o into Eq. (12) and (13), one gets the T dependence of the condensate fraction for $T < T_c^o$

$$\frac{N_o}{N} = 1 - \left(\frac{T}{T_c^o}\right)^3 \quad \ldots(14)$$

The same result can be also obtained by rewriting Eq. (11) as an integral over the energy, in the form -

$$N - N_o = \int_o^\infty \frac{\rho(\varepsilon)\, d\varepsilon}{\varepsilon exp(\beta\varepsilon) - 1} \quad \ldots(15)$$

Putting density of state -

$$\rho(\varepsilon) = \frac{1}{2}(\hbar\omega_{ho})^{-3}\varepsilon^2$$

From eq[n.] (15), we have

$$N - N_o = \int_o^{\infty} \frac{\frac{1}{2}(\hbar\omega_{ho})^{-3}\varepsilon^2 d\varepsilon}{e^{\beta\varepsilon}-1}$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega_{ho})^3}\int_o^{\infty}\frac{\varepsilon^2 d\varepsilon}{e^{\beta\varepsilon}-1}$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega_{ho})^3}\frac{1}{\beta^3}\int_o^{\infty}\frac{y^2 dy}{e^{y}-1} \quad (y = \beta\varepsilon)$$

We have -

$$\int_o^{\infty}\frac{y^2 dy}{e^{y}-1} = \Gamma(3)\,\xi(3) = 2\xi(3)$$

$$N - N_o = \frac{1}{2}\left(\frac{K_\beta T}{\hbar\omega_{ho}}\right)^3 2\xi(3)$$

$$= \xi(3)\left(\frac{K_\beta T}{\hbar\omega_{ho}}\right)^3$$

which is the same as that of equation (12).

On plotting condensate function (N_o/N) as a function of (T/T_c^o), it was found that agreement with experimental result[5] is very poor. Then two corrections were made in order to match the result with the experiment. One is finite size effect and other is role of dimensionality.

The finite size correction to the equation (14) for the condensate fraction can be evaluated analytically by studying the large N-limit[6-13]. The result for ($N_0(T)/N$) is given by -

$$\frac{N_o}{N} = 1 - \left(\frac{T}{T_c^o}\right)^3 - \frac{3\varpi\xi(2)}{2\omega_{ho}[\xi(3)]^{2/3}}\left(\frac{T}{T_c^o}\right)^2 N^{-1/3} \qquad \ldots(16)$$

The result is shown in Table T_1. To the lowest order, finite size effect decrease as $N^{-1/3}$ and depend on the ratio of the arithmetic mean ϖ and geometric (ω_{ho}) average of the

oscillator frequency. For axially symmetric traps, the ratio depends on the deformation parameter

$$\lambda = \omega_z/\omega_l \; as \varpi/\omega_{ho} = (\lambda + 2)/3\lambda^{1/3})$$

Interacting model

As we know that the effects of two body interactions in dilute Bose gas is expected to be significant in the presence of condensate. In this presence of attraction forces, there is narrowing of the peak in the center of the trap and also increase of peak density, one namely discuss the system composed of large number of particles interacting "with repulsive forces.

In the presence of repulsive force, the chemical potential provides an important scale of energy lying between oscillator energy and the critical temperature $\hbar\omega_{ho} < \mu < K_\beta T_c^o$. One defines a very useful parameter η, which is the ratio -

$$\eta = \frac{\mu}{K_\beta T_c^o} = \alpha \left(N^{1/6} \frac{a}{a_{ho}}\right)^{2/5} \quad \ldots(17)$$

This is ratio between the chemical potential calculated at T = 0 in TF approximation and the critical temperature for non-interacting particle in the same trap.

$$\alpha = \frac{(15)^{2/5} \, [\xi(3)]^{1/3}}{2} \cong 1.57$$

α is a numerical coefficient (N a/a_{ho}) accounts for the effects of two body interactions. If one uses the typical values of the parameters of the current experiments, one finds that η ranges from 0.35 to 0.40. Thus, one expects that there is an interaction effects of T_c^o. Using the semi-classical results[14], one can calculate the number of atoms out of the condensate.

$$N_T = \int \frac{d\bar{r}d\bar{p}}{(2\pi\hbar)^3} \exp\left[\frac{[\frac{p^2}{2m} + V_{eff}(r) - \mu]}{K_\beta T} - 1\right]^{-1} \quad \ldots(18)$$

Using the Thomas-Fermi approximation for the effective mean field potential, one can obtain the result[14] -

$$\frac{N_o}{N} = 1 - t^3 - \frac{\xi(2)}{\xi(3)} \eta t^2 (1 - t^3)^{2/5} \quad \ldots(19)$$

This equation is valid to lowest order in the interaction parameter η. t is the reduced

temperature $t = (T / T_c^o)$.

RESULTS AND DISCUSSION

In this paper, we have evaluated the condensate fraction (N_o/N) as a function of (T/T_c^o) using the theoretical formalism developed by F Dalfovo et al.[14] In Tables 1 and 2, we have shown the condensate fraction (N_o/N) for interacting and non- interacting particles in the different traps. For interacting model, we have taken the self-consistent mean field calculation within Popov approximation for $N = 5\times10^4$ rubidium atoms in a trap with $a/a_{ho} = 5 \times 10^{-3}$ and $\lambda = \sqrt{8}$. For $N = 5 \times 10^7$, sodium atoms in a trap with the value of $a/a_{ho} = 5 \times 10^{-3}$ and $\lambda = 0.05$. The numerical results were compared with the prediction of scaling theory for $\eta = 0.4$. The above calculation indicates (in an explicit way), how very different configuration can given rise to the same thermodynamic behavior, if the corresponding scaling parameter η is the same. It is also interesting to notice that the scaling behavior is reached faster in the presence of two body interactions than for non-interacting particles.

Table 1: Evaluated result of condensate fraction (N_o/N) as a function of (T/T_c^o) for interacting and non-interacting particles for Rb atoms $N = 5 \times 10^4$, $a/a_{ho} = 5.4 \times 10^{-3}$ $\lambda = \sqrt{8}$, $\eta = 0.4$

(T / T_c^o)	(N_o/N)	
	Interacting	**Non-Interacting**
0.1	0.787	0.968
0.2	0.924	0.942
0.25	0.896	0.905
0.30	0.824	0.832
0.35	0.745	0.738
0.40	0.692	0.674
0.45	0.612	0.625
0.50	0.554	0.567
0.55	0.526	0.567
0.60	0.497	0.506
0.65	0.432	0.414
0.70	0.386	0.408
0.75	0.339	0.367
0.80	0.268	0.318

Table 2: Evaluated result of condensate fraction (N_o/N) as a function of (T/T_c^o) for interacting and non-interacting particles for sodium atom $N = 5 \times 10^7$, $a/a_{ho} = 1.7 \times 10^{-3}$, $\lambda = 0.05$, $\eta = 0.4$

(T / T_c^o)	(N_o/N)	
	Interacting	**Non-Interacting**
0.1	0.992	0.982
0.2	0.956	0.946
0.3	0.897	0.875
0.4	0.824	0.806
0.5	0.786	0.765
0.6	0.697	0.669
0.7	0.547	0.525
0.8	0.512	0.452
0.9	0.467	0.408
0.95	0.418	0.382
1.0	0.359	0.339

In Table 3, we have given the evaluated result of the condensate fraction as a function of (T / T_c^o) for axially symmetric trap using equation (16) and then the obtained result is compared with equation (14). The condensate fraction obtained in this way, turns out to be smaller than the thermodynamic limit prediction of equation (14). It appears that finite N makes the system potentially richer, because new interesting regimes can be explored even in case, where there is no real phase transition in the thermodynamic regime. With the numerical calculation, Ketterle and Van Druten[11] found that finite size effects are significant only for small values of N less than about 10^4.

In Table 4, we have shown the evaluated result of condensate fraction (N_o/N) as a function of t (= T / T_c^o) for various values of η (a) $\eta = 0.4$, (b) $\eta = 0.6$ and (c) $\eta = 0$ (non-interacting). Results were compared with (d) Monte Carlo calculation by Kranth[16] and experimental data by Ensher et al[5]. The Monte Carlo simulation has been done with 10,000 atoms interacting through an hard core potential. Monte Carlo result corresponds to $\eta = 0.35$. The experimental data corresponds to the η ranges from 0.39 - 0.45. Monte Carlo result is very much near to mean field prediction of η equal to 0.4. A detailed comparison between Monte Carlo results and mean filed theory has been performed by Holzmann, Krauth and

Maraschewski including the analysis of the density profiles of the gas at different temperatures[17]. Recent calculations[18-20] also confirm the same behavior.

Table 3: Evaluated result of condensate fraction (N_o/N) as a function of (T/T_c^o) for axially symmetric traps using equation (16). The results were compared with equation (14)

(T / T_c^o)	(N_o/N)	
	Using eq. (16) taking finite size effect	**Using eq. (14) without finite size effect**
0.1	0.967	0.982
0.2	0.942	0.956
0.3	0.918	0.942
0.4	0.894	0.908
0.45	0.813	0.832
0.50	0.753	0.765
0.55	0.719	0.735
0.60	0.627	0.644
0.65	0.584	0.622
0.70	0.548	0.608
0.75	0.502	0.554
0.80	0.343	0.468
0.85	0.309	0.365
0.90	0.228	0.242
0.95	0.182	0.204

Table 4: Evaluated result of condensate fraction (N_o/N) as a function of (T/T_c^o) for various values of η (a) η = 0, (b) η = 0.4, (c) η = 0.6 and (d) Monte Carlo result (e) Experimental results

(T / T_c^o)	(N_o/N)				
	η = 0	**η = 0.4**	**η = 0.6**	**Monte carlo results**	**Expt. results**
0.1	0.995	0.942	0.922	0.953	0.939

Cont…

(T / T_c^0)	(N_0/N)				
	η = 0	**η = 0.4**	**η = 0.6**	**Monte carlo results**	**Expt. results**
0.2	0.976	0.910	0.894	0.922	0.916
0.3	0.942	0.886	0.856	0.896	0.890
0.4	0.896	0.823	0.796	0.834	0.842
0.45	0.853	0.785	0.763	0.792	0.788
0.50	0.832	0.712	0.709	0.724	0.709
0.55	0.816	0.695	0.675	0.708	0.686
0.60	0.802	0.638	0.612	0.644	0.642
0.65	0.786	0.602	0.588	0.616	0.619
0.70	0.769	0.556	0.543	0.569	0.573
0.75	0.705	0.514	0.496	0.526	0.532
0.80	0.682	0.448	0.428	0.455	0.431
0.85	0.605	0.366	0.343	0.375	0.374
0.90	0.538	0.298	0.277	0.302	0.308
0.95	0.233	0.124	0.116	0.133	0.122

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