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AN EVALUATION OF CHEMICAL POTENTIAL OF QUASI TWO DIMENSIONAL CONDENSATE IN AN ANISOTROPIC TRAP

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ABSTRACT

Using hybrid variational model and Gaussian variational model, chemical potential μ in the unit of $\hbar\omega_z$ for quasi two-dimensional condensate in an anisotropic traps have been evaluated. Our results for both models are very similar in magnitude and trends. Our evaluated results are also in good agreement with other theoretical workers.

Key words: Hybrid variational model, Gaussian variational model, Chemical Potential, Quasi two-dimensional condensate, Anisotropic trap, Dilute atomic gas.

INTRODUCTION

Bose-Einstein condensation of dilute atomic gases has been achieved in a variety of magnetic and optical dipole force traps with different geometries. There is a considerable interest in studying the properties of these ultracold gases under conditions where the confinement gives a system dimensionality less than 3. Recent experiment in optical lattices have observed the properties of a one-dimensional Tonks gas in which bosons show fermionic properties[1,2]. Many other experiments : phase coherence between lattice wells were observed[3,4], collective excitations of a one-dimensional gas were studied and three – body recombination rates in a correlated 1D degenerate Bose gas were measured[5]. All these experiments were carried out with many individual condensates in a lattice of tightly confining potential tubes formed at the inter-section of two optical standing waves. Tunneling between individual wells was controlled through the beam intensities. A single optical potential well was used to confine a mixture of BEC and Fermi gas where the BEC was found to have a one-dimensional character.

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Other experiments used a one-dimensional lattice of BECs formed by a single standing wave. Each individual condensate was confined to an extreme pancake shaped potential well and had quasi-two-dimensional properties; the tunneling between the well could be controlled by adjusting the intensity of the standing wave ,an oscillating current in an array of Josephson junctions was studied[7], number-squeezed states were created[8] and interference between independent condensates was observed[9]. Two-dimensional Bose-condensates in a single potential were studied[10-12]. However, the new physics in this regime remains to be explored. A two-dimensional Bose gas in a homogenous potential does not undergo Bose-Einstein condensation (BEC). Instead there is a Berenzinskii-Kosterlitz-Thouless transition, which is a topological phase transition mediated by the spontaneous formation of vortex pairs. It is a system which is super fluid even though it does not possess long-range order. This is the usual picture of super-fluidity in three dimensions explained in terms of a macroscopic wave function describing the whole system. A recent theoretical paper[13] discussed the dependence of the condensate coherence length on temperature. It shows that even at low temperature, at a fraction of critical temperature, the coherence length is much smaller than the condensate size due to strong phase fluctuations. Early experiments on the KT-transition were carried out with films of super fluid^{4}He and more recent ones include the observation of quasi condensates in thin layers of spin polarized hydrogen[14-16].

In this paper, we have evaluated the chemical potential of quasi two-dimensional condensate in an anisotropic traps by using two models namely hybrid variational model and Gaussian vibrational model. The results for chemical potential $\left(\frac{\mu}{\hbar\omega_z}\right)$ as a function of radial frequency ω_r (H_z) for these two models are very similar for ω_r = 5 to 50 Hz both in magnitude and trend.

Mathematical formulae used in the calculation

Condensates are usually trapped in harmonic potentials given by –

$$V_{ext} = \frac{m}{2}\omega^2{}_0\Sigma_i\lambda^2{}_i x^2{}_i \qquad \ldots(1)$$

where the $\lambda_i(t)$ denote the trap anisotropies which can in general depend on time. A quasi-two-dimensional trap has $\lambda_z >> \lambda_{xy}$. For large anisotropies the condensate shape along the z direction is very similar to the Gaussian profile of an ideal gas. However, along the weakly confined x and y axes the condensate has parabolic shape characterstic of the hydrodynamic regime. The best description in terms of simple analytical functions is therefore to model the condensate wave function as a hybrid of parabola and Gaussian. Experiments on the condensate expansion in various regimes show the smooth crossover from hydrodynamic expansion to the characteristics of a quasi-two-dimensional gas, which essentially expands like an ideal non-interacting gas. To determine the dynamics of the quasi-two-dimensional condensate, we use a variational method[17] and define the trail wave function –

$$\psi = A_n\,[1-\frac{x^2}{l^2_x}-\frac{y^2}{l_x{}^2}]^{\frac{1}{2}}\ \exp(\frac{-z^2}{2l^2_x})\ \exp\{i(\beta_x x^2+\beta_y y^2+\beta_z z^2)\} \qquad \ldots(2)$$

Where the normalization constant A_n is given by –

$$A^2{}_n = \frac{2}{l_x l_y l_z \pi^{\frac{3}{2}}} \qquad \ldots(3)$$

The condensate width $l_i(t)$ and phase $\beta_i(t)$ parameters are functions of time and their time evaluation completely describe that of the condensate. The condensate profile is at all time restricted to a parabolic shape in the radial plane and a Gaussian shape along the highly compressed axial direction. The Lagrangian density for the nonlinear Schrodinger equation is given by –

$$\alpha = \frac{1}{2}i\hbar(\frac{\partial\psi^*}{\partial t}\psi-\psi^*\frac{\partial\psi}{\partial t})\ +\frac{\hbar^2}{2m}\left|\nabla\psi\right|^2+V_{ext}(r,t)\left|\psi\right|^2+\frac{1}{2}g\left|\psi\right|^4 \qquad \ldots(4)$$

With the nonlinearity parameter $g=\frac{4\pi\hbar^2 a}{m}$, where a is the scattering length, N is the number of atoms in the condensate and m is the atomic mass. After inserting the trial wave function (2) into Equation (4) the corresponding Lagrangian is found through integration $L = \alpha d^3$; the four terms of Eq. (4) lead to –

$$L = L_1 + L_2 + L_3 + L_4$$

$$\frac{\hbar^2}{m}(\frac{\beta^{\circ}_x l^2_x}{3}+\frac{\beta^{\circ}_y l^2_y}{3}+\beta^{\circ}_z l^2_z)+\frac{\hbar^2}{m}(\frac{\beta^2_x l^2_x}{3}+\frac{\beta^2_y l^2_y}{3}+\beta^2_z l^2_z+\frac{1}{4l^2_z})$$

$$\mathrm{dd}+\frac{m}{4}\left(\frac{\omega_x{}^2 l^2{}_x}{3}+\frac{\omega_y{}^2 l^2{}_y}{3}+\omega^2{}_z l^2{}_z\right)+\frac{\sqrt{2}}{3 l_x l_y l_z \pi^{\frac{3}{2}}} gN \qquad \ldots(5)$$

Where, we have obtained the 'quantum pressure' term[18] for the x and y directions (where this term is divergent due to the sharp boundaries of the condensate wave function in the hydrodynamic regime) but retained it for the z direction where the condensate assumes the Gaussian shape of an ideal non-interacting gas (as the term proportional to $\frac{1}{l_z^2}$). The quantum pressure term is crucial in describing the dynamics. The total energy per particle E_{tot} and the chemical potential μ are given by –

$$E_{tot} = E_{kin}+ E_{pot}+ E_{int}\,, \quad \mu= E_{kin}+ E_{pot}+ E_{int} \qquad \ldots(6)$$

where E_{kin}, E_{pot} and E_{int} are the kinetic, potential and interaction energy given in the last term of the Lagrangian (5), respectively. The Euler Lagrange equations –

$$\frac{d}{dt}\frac{\partial L}{\partial l^{\circ}{}_i}=\frac{\partial L}{\partial l_i} \qquad \frac{d}{dt}\frac{\partial L}{\partial \beta^{\circ}{}_i}=\frac{\partial L}{\partial \beta_i} \qquad \ldots(7)$$

Yield the dynamic equation for the condensate width l_i and phase β_i. We find for the widths –

$$l_i = \frac{2\hbar}{m}\beta_i l_i \qquad \ldots(8)$$

After differentiating Eqs. (8) with respect to time, one can express the resulting equation in terms of the l_i alone –

$$\ddot{l}_i = -\omega^2{}_i(t) l_i + \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \frac{gN}{m}\frac{1}{l_i l_x l_y l_z}\left(1-\frac{2}{\delta_{iz}}\right)+\frac{\hbar^2}{m^2}\frac{1}{l^3{}_i}\delta_{iz} \qquad \ldots(9)$$

Where δ_{iz} – 1 for I = z and 0 otherwise. It is convenient to express the above equation in dimensionless quantities, so one introduces dimensionless time τ and width d_i defined by –

$$d_i = \frac{l_i}{a_0}, \tau = t\omega_0 \qquad \ldots(10)$$

where $a_0 = \sqrt{\hbar/(m\omega_0)}$ is the harmonic oscillator length. In terms of quantities Eq. (9) can be rewritten as –

$$\ddot{d}_i = -\lambda_i^2(t)d_i + (\frac{C_p}{d_i d_x d_y d_z})(1 - \frac{2}{3}\delta_{iz}) + \frac{1}{d^3_x}\delta_{iz} \qquad \ldots(11)$$

where the constant $C_p = 8\sqrt{2/\pi}(\frac{a}{a_0})N$. To find ground state of Eqs. (11) one has to set the left side equal to zero and solve the remaining coupled nonlinear equations :

$$d_{i0}^{\ 2}\lambda_{i0}^{\ 2} = (\frac{C_p}{d_{x0}d_{y0}d_{z0}})(1 - \frac{2}{3}\delta_{iz}) + \frac{1}{d_{z0}^{\ 2}}\delta_{iz} \qquad \ldots(12)$$

This cannot be done analytically, but it is straightforward to find a numerical solution. The $\lambda_{io} = \lambda_i$ (0), $i = x, y, z$ are defined as trap anisotropies at time t = 0 when the condensate is in the ground state. The $d_{i0} = d_i(0)$ are the ground state solution of Eq. (11), i.e. the solutions for the condensate widths d_i when the time derivative is set to zero.

After some algebra and using various symmetries the three coupled equations can be reduced to one polynomial equation. Introducing new dimensionless units D_i, defined as the ground state condensate widths l_{i0} normalized by the axial harmonic oscillator length a_z, ie., $D_i = l_{i0}/a_z$, the poly-nomial equation can be written as –

$$\gamma^2 = \frac{1}{3}(\frac{C_p\lambda_{xo}\lambda_{yo}}{\lambda^{\frac{3}{2}}_{\ z}})^{\frac{1}{2}}\gamma^3 + 1 \qquad \ldots(13)$$

Where $D_z = \gamma^2$. There is only one real and positive solution to this equation. For the x and y widths, we find –

$$D_x = (\frac{C_p}{D_x}\frac{C_p\lambda_{yo}\lambda^{5/2}_{\ zo}}{\lambda^3_{\ xo}}), D_y = D_x\frac{\lambda_{xo}}{\lambda_{yo}} \qquad \ldots(14)$$

Now, one examines the case where the anisotropy becomes very large. A solution to Eq. (13) is then given by neglecting the first term on the R.H.S. and solving the remaining equation. We find that $\gamma^2 = D_z = 1$ and thus the approximate solution is given by the axial harmonic oscillator length.

$$I_{zo} = \sqrt{\frac{\hbar}{m\omega_z}} \qquad \ldots(15)$$

It is the minimum width the condensate shape can attain and it is also the solution for the width of a non interacting gas. For this reason the gas along the Z direction is said to have the characteristics of an ideal non interacting gas. One will see that this also applies to the expansion and the collective excitation of the gas which become identical to those of the ideal gas in the limit of large anisotropies.

It is interesting to examine the range of validity of this approximation and find an estimate of the error. Demanding that the first term on the R. H. S. of Eq. (13) is much smaller than the second, one sees that the error of the deviation from the ideal gas solution scales with the relation of $N\, \lambda_{xo}\lambda_{yo}/\lambda^{3/2}_{zo}$. The 2D regime can be reached by either decreasing the number or increasing the axial frequency. Now, one can calculates the chemical potential from Eq. (6) and the terms of the Lagrangian (5) and obtain after some algebra –

$$\bar{\mu} = \frac{1}{2} m\omega^2_{x} I^2_{xo}, \bar{\mu} = \mu - \frac{\hbar\omega_z}{2} \qquad \ldots(16)$$

Where one has used $\omega^2_{xo} I^2_{xo} = \omega^2_{yo} I^2_{yo}$, The expression for I_{zo} (Eq. (15)) and other symmetries of Eq. (12). One finds that the relation $I_{io} = \sqrt{2\bar{\mu}/m\omega^2_i}$, I = x, y is similar to that of a hydrodynamic gas[19] only that for the quasi-two-dimensional gas one uses the chemical potential shifted by an amount $\hbar\omega_z/2$ to calculate the radial width. Inserting solution (15) for the axial width into Eq. (14) one obtains explicit expressions for the radial width –

$$I^2_{xo} = a^2_0 (8\sqrt{\frac{2}{\pi}} N \frac{a}{a_0} \frac{\lambda_{yo}\lambda^{\frac{1}{2}}_{zo}}{\lambda^3_{xo}})^{\frac{1}{2}} \qquad \ldots(17)$$

and after substituting into Eq. (16) one obtains for the chemical potential –

$$\mu = \frac{\hbar\omega}{2}[1 + (8\sqrt{\frac{2}{\pi}} N \frac{a}{a_0} \frac{\lambda_{xo}\lambda_{yo}}{\lambda^{\frac{3}{2}}_{zo}})^{\frac{1}{2}}] \qquad \ldots(18)$$

Here a is the Bohr radius. This expression shows that the chemical potential tends towards $\frac{\hbar\omega_z}{2}$, the harmonic oscillator ground state energy, which is the energy per particle and also the chemical potential of the ideal non-interacting gas. The deviation from this value is small for a quasi-two-dimensional gas, and interestingly, given by the same value as the correction to the axial condensate width of Eq. (13). It is proportional to the square of the radial condensate width would also be zero which is neither possible nor self-consistent. The expression for Q2D chemical potential should be compared to that of 3D hydrodynamic gas μ_{3D} for which one obtains[19] from after some rearrangements.

$$\mu_{3D} = \frac{\hbar\omega_z}{2}(15N\frac{a}{a_o}\frac{\lambda_{xo}\lambda_{yo}}{\lambda^{\frac{3}{2}}_{zo}})^{\frac{2}{5}} \qquad \ldots(19)$$

One observed that this expression tends towards zero for N → 0 and the power law is also different from the Q2D expression (18). Similarly one can impose the condition $\mu \prec \hbar\omega_z$ on the expression of the Q2D chemical potential in Eq. (18) and one finds for the maximum number of atoms to achieve 2D for a given trap geometry.

$$N \prec C\sqrt{\frac{\hbar}{ma^2}}\sqrt{\frac{\omega^3_z}{\omega^2_x\omega^2_y}} \qquad \ldots(20)$$

Where the constant $C = \sqrt{\frac{32}{225}} for \mu_{3D} \prec \hbar\omega_z and\ C = \sqrt{\frac{\pi}{256}} for \mu \prec \hbar\omega_z$.

RESULTS AND DISCUSSION

In this paper, we have evaluated the chemical potential of quasi-two-dimensional condensate in anisotropic traps. The evaluation has been performed using the theoretical model developed by Hechenblaikner et al.[21] The Chemical potentials were calculated for two different models one is Hybrid variational model and other is Gaussian variational model as a function of increasing radial trap frequency ω_r. The axial frequency remains constant at $\frac{\omega_c}{2\pi} = 2.2\ \mathrm{KH_z}$ and the number of atoms is taken to be N = 8 x 10^4. Our theoretically evaluated results show that chemical potential increases with radial frequency ω_r. For very small values of ω_r the anisotropy A $\left(\lambda_z/\lambda_x\right)$ is very high and the chemical potential approaches $\mu \approx \frac{\hbar\omega_z}{2}$. Gradually increasing the radial trap frequency the anisotropy has been reduced and chemical potential increases. The results are shown in Table 1.

Table 1: Evaluated results of chemical potential $\mu/\hbar\omega_z$ (in the unit of $\hbar\omega_z$) as a function of radial trap frequency. The calculation is done with the help of (a) Hybrid variational model (b) Gaussian variational model

Radial frequency ω_r (H_z)	Chemical Potential μ (in the unit of $\hbar\omega_z$)	
	(a)	(b)
5	0.52	0.55
10	0.58	0.63
15	0.65	0.69
20	0.68	0.73
25	0.72	0.75
30	0.74	0.80
32	0.82	0.85
34	0.86	0.88
36	0.87	0.94
38	0.92	0.98
40	0.95	1.02
45	1.08	1.10
50	1.12	1.15

In (a), we have shown the result of variational hybrid model and in (b) we have shown the results of Gaussian variational model. In Gaussian variational model[20,21], the trial wave function taken in all spatial direction is Gaussian. The results along with the trends of both the models are the same. Some recent calculations[22-25] also reveals the same facts.

REFERENCES

1. B. Paredes, A. Windera, V. Murg, O. Mandel, S. Folling, I. Cirac, G. Shlyapnikov, T. Hansch and I. Bloch , Nature (London), **429**, 277 (2004).
2. T. Kinoshita, T. Wenger and D. S. Weiss, Science, **305**, 1125 (2004).
3. M. Greiner, I. Bloch, O. Mandel, T. W. Hansch and T. Esslinger, Phys. Rev. Lett. (PRL), **87**, 160405 (2001).

4. H. Moritz, T. Stferle, M. Kohl and T. Esslinger, Phys. Rev. Lett. (PRL), **91**, 250402 (2003).
5. B.L. Tolra, K. M. O'Hara, J. H. Kuckans, W. D. Phillips, S. L. Rolston and J. V. Porto, Phys. Rev. Lett. (PRL), **92**, 190401 (2004).
6. F. Schreck, L. Khaykovich, K. L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles and C. Salomon, Phys. Rev. Lett. (PRL), **87**, 080403 (2001).
7. F. S. Cataliotti, S. Burger, C. Fort, F. Minardi, A. Trombettoni, A. Smerzi and M. Inguscio, Science, **293**, 843 (2001).
8. C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda and M. A. Kasevich, Science **291**, 2386 (2001).
9. Z. Hadzibabic, S. Stock, B. Battelier, V. Bretin and J. Dailbard, Phys. Rev. Lett. (PRL), 93, 180403 (2004).
10. A. Gorlitz, A.L. Safonow, S. Stringari, H. Michinel and P. Zoller, Phys. Rev. Lett. (PRL), **87**, 130402 (2001).
11. D. Rychtarik, B. Engeser, H. C. Nageri and R. Grimm, Phys. Rev. Lett. (PRL), **92**, 173003 (2004).
12. N. L. Smith, W. Heathcote, G. Hechenblaikner, E. Nugent and C. J. Foot, J. Phys. Condens. Matter, **14**, 1021 (2002).
13. C. Gies and D. A. W. Hutchinson, Phys. Rev., **A70**, 043606 (2004),
14. D. J. Bishop and J. D. Reppy, Phys. Rev. Lett. (PRL), **92**, 180631 (2004).
15. J. M. Kosterlitz and D. J. Thouless, J. Phys., **C 6**, 1181 (1973).
16. A. L. Safonow, S. A. Vasilyev, I. S. Yasnikov I. I. Lukashevich and S. Jaakkola, Phys. Rev. Lett. (PRL), **81**, 4545 (1998).
17. V. M. Perez-Garcia, H. Michinel, J. I. Cirac, M. Lewenstein and P. Zoller, Phys. Rev. Lett. (PRL), **77**, 5320 (1996).
18. S. Stringari, Phys. Rev. Lett. (PRL), **77**, 2360 (1996).
19. F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys., **71**, 463 (1999).
20. Y. Castin and R. Dum, Phys. Rev. Lett. (PRL), **77**, 5315 (1996).
21. G. Hechenblaikner, N. L. Smith, C. J. Foot, W. Heathcote and E. Nugent, Phys. Rev., **A71**, 013064 (2004).

22. G. Hechenblaikner, J. M. Krueger and C. J. Foot, Phys. Rev., **A72**, 014604 (2005).
23. G. Watanabe, S. Andrew, G. Baym and C. J. Pethick, Phys. Rev., **A74**, 063621 (2006).
24. S. R. Mishra, S. P. Ram and A. Banerge, Pramana –J. Phys., **68**, 913 (2007).
25. F. Malet, T. Kristensen, S. M. Reimann and G. M. Kavoulakin, Phys. Rev., **A83**, 033628 (2011).

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